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Widths of the Hall Conductance Plateaus

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Abstract We study the charge transport of the noninteracting electron gas in a two-dimensional quantum Hall system with Anderson-type impurities at zero temperature. We prove that there exist localized states of the bulk order in the disordered-broadened Landau bands whose energies are smaller than a certain value determined by the strength of the uniform magnetic field. We also prove that, when the Fermi level lies in the localization regime, the Hall conductance is quantized to the desired integer and shows the plateau of the bulk order for varying the filling factor of the electrons rather than the Fermi level.

Keywords Quantum Hall effect · Landau Hamiltonian · Strong magnetic field · Anderson localization · Hall conductance plateaus

1 Introduction

The two most remarkable facts of the integral quantum Hall effect [1] are the integrality of the Hall conductance and its robustness for varying the parameters such as the filling factor of the electrons and the strength of the disorder. The integrality is explained by the topological nature [2, 3] of the Hall conductance. The constancy of the Hall conductance is due to the Anderson localization of the wavefunctions of the electrons [4].

First of all we shall survey recent mathematical analysis of the quantum Hall effect. As for justification of the conductance formula leading to the topological invariant, satisfactory results have been obtained in the recent papers within the linear response approximation or an adiabatic limit of slowly applying an electric field [5–9]. Avron, Seiler and Yaffe [5] proved that a flux averaged charge transport[1] is quantized to an integer in the adiabatic limit under the assumption of a nonvanishing spectral gap above a non-degenerate ground state

[1] This is a non-trivial charge transport which is intrinsically different from the response to a static external field.

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for a finite-volume interacting electron gas. In [6], a static electric field with a regularized boundary condition was used as an external force to derive an electric current for a finite-volume interacting electron gas under the assumption of a nonvanishing spectral gap above the sector of the ground state(s). The resulting Hall conductance is equal to the universal conductance multiplied by the filling factor of the electrons in the infinite-volume limit. When the Fermi level lies in a spectral gap for a noninteracting electron gas on the whole plane $\mathbf{R}^2$, Elgart and Schlein [7] justified the Hall conductance formula which is written in terms of switch functions in the adiabatic limit. This formula was first introduced by Avron, Seiler and Simon [10, 11]. Without relying on the gap assumption, a general conductance formula was obtained for finite-volume interacting electron gases [8]. For the whole plane, Bouclet, Germinet, Klein and Schenker [9] obtained a Hall conductance formula for a random noninteracting electron gas with translation ergodicity under the assumption that the Fermi level falls into a localization regime.

As to the localization and the related conductance plateaus, we refer only to a class of noninteracting electron gases because the localization of interacting electrons is still an unsolved problem. The existence of the localization at the edges of the disordered-broadened Landau bands was proved within a single-band approximation [12–14], for a sufficiently strong magnetic field [15–17], or for a low density of the electrons at the band edges [18]. The existence of the quantized Hall conductance plateaus was first proved by Kunz [19] under assumptions on a linear response formula of the conductance and on the band edge localization. The latter assumption on the localization can be removed for a tight-binding model. Namely, the constancy of the quantized Hall conductance was proved within the tight-binding approximation for varying the Fermi level [20, 21], or the strength of the potential [22]. We should remark that, without relying on the translation ergodicity of the Hamiltonian, Elgart, Graf and Schenker [23] proved the constancy of the quantized Hall conductance for a tight-binding case. For continuous models, Nakamura and Bellissard [24] proved that the states at the bottom of the spectrum do not contribute to the Hall conductance. Quite recently, Germinet, Klein and Schenker [25] proved that the Hall conductance formula [10, 11] which is written in terms of switch functions shows a plateau for a random Landau Hamiltonian with translation ergodicity. In order to determine the integer of the quantized value of the Hall conductance, they further required the condition that the disordered-broadened Landau bands are disjoint, i.e., there exists a nonvanishing spectral gap between two neighboring Landau bands. However, the existence of the localized states at the band edges does not necessarily implies the appearance of the Hall conductance plateaus for varying the filling factor of the electrons because the density of the localized states may be vanishing in the infinite volume. In order to show the existence of such plateaus, we need to prove the existence of localized states of the bulk order. In passing, we remark that Wang [26] obtained the asymptotic expansion for the density of states in the large magnetic field limit.[2]

In this paper, we focus on the issue of proving the bulk order plateaus, and consider a noninteracting electron gas with Anderson-type impurities in a magnetic field in two dimensions at zero temperature. The centers of the bumps of the impurities form the triangular lattice. First we prove that there exist localized states of the bulk order in the disordered-broadened

[2]In general, an asymptotic series does not give us any information for a fixed finite value of the parameter because the asymptotic series is not necessarily convergent. See, for example, Sect. XII.3 of the book [27]. Thus the result of [26] dose not imply the existence of localized states of the bulk order for a fixed finite value of the magnetic field. See also the recent paper [28] for the difficulty of obtaining a lower bound for the density of states.

Landau bands whose energies are smaller than a certain value determined by the strength of the magnetic field. In order to obtain the Hall conductance as a linear response coefficient to an external electric field, we apply a time-dependent vector potential $\mathbf{A}_{\mathrm{ex}}(t) = (0, \alpha(t))$, where the function $\alpha(t)$ of time t is given by (2.18) in the next section. For $t \in [-T, 0]$ with a large positive T, the corresponding electric field is adiabatically switched on, and for $t \geq 0$, the electric field becomes $(0, F)$ with the constant strength F. First we consider the finite, isolated system of an $L_x \times L_y$ rectangular box, and impose periodic boundary conditions for the wavefunctions with the help of the magnetic translation (2.17), and then we take the infinite-volume limit. The explicit expression of the conductance formula which we will use is given in Ref. [8]. We should remark that, when the system is translationally invariant, the constant Hall current flows on the torus without dissipation of energy as in Ref. [8]. We prove that, when the Fermi level lies in the localization regime, the Hall conductance is quantized to the desired integer and shows the plateau of the bulk order for varying the filling factor of the electrons. In our approach, we require neither the disjoint condition for the Landau bands nor the translation ergodicity[3] of the Hamiltonian which were assumed in [25] as mentioned above. Instead of these condition, we need the "covering condition" that the whole plane $\mathbf{R}^2$ is covered by the supports of the bumps of the impurity potentials so that the sum of the bumps is strictly positive on the whole plane $\mathbf{R}^2$. This "covering condition" is not required in [25].

The present paper is organized as follows. In Sect. 2, we describe the model, and state our main theorems. As preliminaries, we study the spectrum of the Hamiltonian without the random potential in Sect. 3 and the site percolation on the triangular lattice of the impurities in Sect. 4. In Sect. 5, we obtain a decay bound for the resolvent (Green function) of a finite volume. This bound becomes the initial data for the multi-scale analysis [29–31] which is given in Sect. 6. In order to prove constancy of the Hall conductance, we further need the fractional moment bound [32] for the resolvent. The bound is given in Sect. 7. As preliminaries for proving the integrality and constancy of the Hall conductance, we study the finite volume Hall conductance in Sect. 8. The integrality of the Hall conductance is proved within the framework of "noncommutative geometry" [10, 11, 20, 21] in Sect. 9, and the constancy is proved by using the homotopy argument [20, 22] in Sect. 10. The widths of the Hall conductance plateaus and the corrections to the linear response formula are estimated in Sects. 11 and 12, respectively. Appendices A–H are devoted to technical estimates. The standard Hall conductance formula which is given in Sect. 9 is written in terms of the position operator of the electron. In Appendix I, we give a proof that this Hall conductance is equal to another Hall conductance [10, 11] which is written in terms of switch functions for a class of continuous models.

2 Model and the Main Results

Consider a two-dimensional electron system with Anderson-type impurities in a uniform magnetic field $(0, 0, B)$ perpendicular to the x–y plane in which the electron is confined. For simplicity we assume that the electron does not have the spin degrees of freedom. The Hamiltonian is given by

$$H_\omega = H_0 + V_\omega \tag{2.1}$$

[3] In a generic, realistic situation that there exist one- or two-dimensional objects such as dislocations in crystals and interfaces in semiconductors, we cannot expect that the system has translation ergodicity.

with the unperturbed Hamiltonian,

$$H_0 = \frac{1}{2m_e}(\mathbf{p} + e\mathbf{A})^2 + V_0, \tag{2.2}$$

and with a random potential V_ω, where $\mathbf{p} := -i\hbar\nabla$ with the Planck constant $\hbar$, and $-e$ and m_e are, respectively, the charge of electron and the mass of electron; $\mathbf{A}$ and V_0 are, respectively, a vector potential and an electrostatic potential. The system is defined on a rectangular box

$$\Lambda^{\mathrm{sys}} := [-L_x/2, L_x/2] \times [-L_y/2, L_y/2] \subset \mathbf{R}^2 \tag{2.3}$$

with the periodic boundary conditions. The vector potential $\mathbf{A} = (A_x, A_y)$ consists of two parts as $\mathbf{A} = \mathbf{A}_{\mathrm{P}} + \mathbf{A}_0$, where $\mathbf{A}_0(\mathbf{r}) = (-By, 0)$ which gives the uniform magnetic field and the vector potential $\mathbf{A}_{\mathrm{P}}$ satisfies the periodic boundary condition,

$$\mathbf{A}_{\mathrm{P}}(x + L_x, y) = \mathbf{A}_{\mathrm{P}}(x, y + L_y) = \mathbf{A}_{\mathrm{P}}(x, y). \tag{2.4}$$

This condition for $\mathbf{A}_{\mathrm{P}}$ implies that the corresponding magnetic flux piercing the rectangular box Λ^{sys} is vanishing. Therefore the total magnetic flux is given by BL_xL_y from the vector potential $\mathbf{A}_0$ only. We assume that the components of the vector potential $\mathbf{A}_{\mathrm{P}}$ are continuously differentiable on $\mathbf{R}^2$. Further we assume that the electrostatic potential V_0 satisfies the periodic boundary condition,

$$V_0(x + L_x, y) = V_0(x, y + L_y) = V_0(x, y), \tag{2.5}$$

and $\|V_0^+\|_\infty + \|V_0^-\|_\infty \le v_0 < \infty$ with some positive constant v_0 which is independent of the system sizes L_x, L_y. Here $V_0^\pm = \max\{\pm V_0, 0\}$. As a random potential V_ω, we consider an Anderson-type impurity potential,

$$V_\omega(\mathbf{r}) = \sum_{\mathbf{z}\in\mathbf{L}^2} \lambda_{\mathbf{z}}(\omega) u(\mathbf{r} - \mathbf{z}), \tag{2.6}$$

for $\mathbf{r} := (x, y) \in \mathbf{R}^2$. The constants $\{\lambda_{\mathbf{z}}(\omega) \mid \mathbf{z} \in \mathbf{L}^2\}$ form a family of independent, identically distributed random variables on the two-dimensional triangular lattice $\mathbf{L}^2 \subset \mathbf{R}^2$ with the lattice constant $a > 0$. The common distribution of the random variables has a density $g \ge 0$ which has compact support, i.e., $\operatorname{supp} g \subset [\lambda_{\min}, \lambda_{\max}]$ with $\lambda_{\min} < 0 < \lambda_{\max}$. Further the density g satisfies the following conditions:

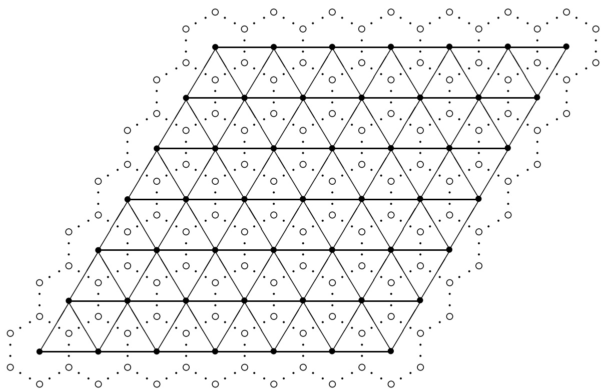
$$g \in L^\infty(\mathbf{R}) \cap C(\mathbf{R}) \quad \text{and} \quad \int_{-\lambda_-}^{\lambda_+} g(\lambda) d\lambda > 1/2 \tag{2.7}$$

with two positive numbers λ_+ and λ_-. We consider two cases: (i) a small λ_- and (ii) a small λ_+. We assume that the condition (2.7) holds for both of the two cases. If the density g is an even function of λ and is concentrated near $\lambda = 0$, this requirement holds. We take

$$\mathbf{L}^2 = \{\mathbf{z} = m\mathbf{a}_1 + n\mathbf{a}_2 \mid (m, n) \in \mathbf{Z}^2\} \tag{2.8}$$

with the two primitive translation vectors, $\mathbf{a}_1 = (a, 0)$ and $\mathbf{a}_2 = (a/2, \sqrt{3}a/2)$. The triangular lattice is embedded in $\mathbf{R}^2$ such that each face is an equilateral triangle as described in Fig. 1. We also consider its dual, hexagonal lattice which is defined as follows. Choose a vertex of the dual lattice at the center of gravity of each triangle, i.e., the intersection of the

Fig. 1 The parallelogram in the triangular lattice with its dual, hexagonal lattice



bisectors of the sides of the triangle. For the edges of the dual lattice, take the line segments along these same bisectors, and connecting the centers of gravity of adjacent triangles.

We assume that the bump u of the single-site potential in (2.6) satisfies the following conditions: $0 \le u \in L^\infty(\mathbf{R}^2)$,

$$u(\mathbf{r}) = 0 \quad \text{for } |\mathbf{r}| \ge r_u \text{ with a constant } r_u \in (\sqrt{3}a/3, \sqrt{3}a/2), \tag{2.9}$$

and

$$u(\mathbf{r}) \ge u_0 > 0 \quad \text{for } \mathbf{r} \text{ in the face of the hexagon with the center } \mathbf{r} = 0 \text{ of gravity.} \tag{2.10}$$

Here u_0 is a positive constant. The first condition (2.9) implies that the single-site potentials u has compact support, and overlap with only nearest neighbor u. The next condition (2.10) implies that the whole space $\mathbf{R}^2$ is covered by the supports of the bumps $\{u(\cdot - \mathbf{z})\}_{\mathbf{z}\in\mathbf{L}^2}$ of the impurities so that

$$\sum_{\mathbf{z}\in\mathbf{L}^2} u(\mathbf{r} - \mathbf{z}) \ge u_0 \quad \text{for any } \mathbf{r} \in \mathbf{R}^2. \tag{2.11}$$

We should remark that this "covering condition" is needed for estimating the number of the localized states and for applying the fractional moment method [32], which yields a decaying bound for the resolvent.

Clearly the random potential V_ω of (2.6) does not necessarily satisfy the periodic boundary condition,

$$V_\omega(x + L_x, y) = V_\omega(x, y + L_y) = V_\omega(x, y), \tag{2.12}$$

without a special relation between the lattice constant a and the system sizes L_x, L_y. Therefore we will replace the random potential V_ω with $\tilde{V}_\omega$ which is slightly different from V_ω in a neighborhood of the boundaries so that $\tilde{V}_\omega$ satisfies the periodic boundary condition (2.12). Before proceeding further, we check that the boundary effect due to this procedure is almost negligible and does not affect the following argument. Write

$$L_x^{\mathrm{P}}/2 = N_x a \quad \text{and} \quad L_y^{\mathrm{P}}/2 = N_y \cdot \sqrt{3}a/2 \quad \text{with positive integers } N_x, N_y. \tag{2.13}$$

When we take the sizes to be $L_x = L_x^{\mathrm{P}}$ and $L_y = L_y^{\mathrm{P}}$, the periodic boundary condition is automatically satisfied without replacing the random potential. However, for a given lattice constant a, the sizes do not necessarily satisfy the flux quantization condition, $L_x^{\mathrm{P}} L_y^{\mathrm{P}} = 2\pi M \ell_B^2$, which we need in the following argument. Here M is a positive integer, and ℓ_B is the so-called magnetic length defined as $\ell_B := \sqrt{\hbar/(eB)}$. In a generic situation, we have

$$2\pi M \ell_B^2 < L_x^{\mathrm{P}} L_y^{\mathrm{P}} < 2\pi (M+1)\ell_B^2 \quad \text{with some positive integer } M. \tag{2.14}$$

In order to recover the flux quantization condition, we change the size a little bit in the y direction. Namely we choose the sizes as $L_x = L_x^{\mathrm{P}}$ and $L_y = L_y^{\mathrm{P}} - \delta L_y$ with a small δL_y. Substituting this into $L_x L_y = 2\pi M \ell_B^2$, we have

$$0 < \delta L_y < 2\pi \ell_B^2 / L_x. \tag{2.15}$$

Notice $L_y < L_y^{\mathrm{P}}$, and consider the triangles which overlap with the upper boundary of Λ^{sys}. We replace these equilateral triangles with the isosceles triangles of the height $(\sqrt{3}a/2 - \delta L_y)$. From the above bound (2.15) for δL_y, the height of the isosceles triangles is slightly shorter than the height $\sqrt{3}a/2$ of the equilateral triangles for a sufficiently large L_x. In the same way as in (2.6), we put the impurity potentials on the center of the gravity of the isosceles triangles. Because of the bound (2.15) for δL_y, the effect of this procedure is negligibly small for a sufficiently large size L_x. When the boundary effect plays an essential role as in Sect. 9 below, we denote by $\tilde{V}_{\omega,\Lambda}$ this resulting random potential for a region Λ. Otherwise, we will often use the same notation V_ω for short.

When $\mathbf{A}_{\mathrm{P}} = 0$, we require the differentiability, $V = V_0 + V_\omega \in C^2$, in addition to the above conditions, in order to obtain the exponential decay bound for the resolvent $(H_\omega - z)^{-1}$ in Appendix D.2.

As mentioned above, we require the flux quantization condition, $L_x L_y = 2\pi M \ell_B^2$, with a sufficiently large positive integer M. The number M is exactly equal to the number of the states in a single Landau level of the single-electron Hamiltonian in the simple uniform magnetic field with no electrostatic potential. This condition $L_x L_y = 2\pi M \ell_B^2$ for the sizes L_x, L_y is convenient for imposing the following periodic boundary conditions: For an electron wavefunction φ, we impose periodic boundary conditions,

$$t^{(x)}(L_x)\varphi(\mathbf{r}) = \varphi(\mathbf{r}) \quad \text{and} \quad t^{(y)}(L_y)\varphi(\mathbf{r}) = \varphi(\mathbf{r}), \tag{2.16}$$

where $t^{(x)}(\cdots)$ and $t^{(y)}(\cdots)$ are magnetic translation operators [33, 34] defined as

$$\begin{aligned} t^{(x)}(x') f(x,y) &= f(x - x', y), \\ t^{(y)}(y') f(x,y) &= \exp[iy'x/\ell_B^2] f(x, y - y') \end{aligned} \tag{2.17}$$

for a function f on $\mathbf{R}^2$.

In order to measure the conductance, we introduce the time-dependent vector field $\mathbf{A}_{\mathrm{ex}}(t) = (0, \alpha(t))$ with

$$\alpha(t) = -Ft \times \begin{cases} e^{\eta t}, & t \le 0; \\ 1, & t > 0. \end{cases} \tag{2.18}$$

Here F is the strength of the electric field, and $\eta > 0$ is a small adiabatic parameter. The y-component of the corresponding external electric field is given by

$$E_{\mathrm{ex},y}(t) = -\frac{\partial}{\partial t}\alpha(t) = \begin{cases} F(1+\eta t)e^{\eta t}, & t \le 0; \\ F, & t > 0. \end{cases} \tag{2.19}$$

The time-dependent Hamiltonian is given by

$$H_\omega(t) = \frac{1}{2m_e}[\mathbf{p} + e\mathbf{A} + e\mathbf{A}_{\mathrm{ex}}(t)]^2 + V_0 + V_\omega. \tag{2.20}$$

The velocity operator is

$$\mathbf{v}(t) = (v_x(t), v_y(t)) = \frac{1}{m_e}[\mathbf{p} + e\mathbf{A} + e\mathbf{A}_{\mathrm{ex}}(t)]. \tag{2.21}$$

Let $U(t, t_0)$ be the time evolution operator. We choose the initial time $t = t_0 = -T$ with a large $T > 0$. Then the total current density is given by

$$\mathbf{j}_{\mathrm{tot}}(t) = (j_{\mathrm{tot},x}(t), j_{\mathrm{tot},y}(t)) = -\frac{e}{L_x L_y} \mathrm{Tr}\, U^\dagger(t, t_0)\mathbf{v}(t)U(t, t_0)P_{\mathrm{F}} \quad \text{for } t \ge 0, \tag{2.22}$$

where P_{F} is the projection on energies smaller than the Fermi energy E_{F}. This total current density is decomposed into [8] the initial current density $\mathbf{j}_0$ and the induced current density $\mathbf{j}_{\mathrm{ind}}(t)$ as $\mathbf{j}_{\mathrm{tot}}(t) = \mathbf{j}_0 + \mathbf{j}_{\mathrm{ind}}(t)$. Here the initial current density $\mathbf{j}_0$ is given by

$$\mathbf{j}_0 = -\frac{e}{L_x L_y} \mathrm{Tr}\, \frac{1}{m_e}[\mathbf{p} + e\mathbf{A}]P_{\mathrm{F}}. \tag{2.23}$$

Further the induced current density $\mathbf{j}_{\mathrm{ind}}(t)$ is decomposed into the linear part and the nonlinear part in the strength F as

$$\mathbf{j}_{\mathrm{ind}}(t) = (\sigma_{\mathrm{tot},xy}(t), \sigma_{\mathrm{tot},yy}(t))F + \mathbf{j}'_{\mathrm{ind}}(t) \quad \text{with } \mathbf{j}'_{\mathrm{ind}}(t) = o(F), \tag{2.24}$$

where the coefficients, $\sigma_{\mathrm{tot},sy}(t)$ for $s = x, y$, of the linear term are the total conductance which are written

$$\sigma_{\mathrm{tot},sy}(t) = \sigma_{sy} + \delta\sigma_{sy}(t) \tag{2.25}$$

with the small corrections, $\delta\sigma_{sy}(t)$, due to the initial adiabatic process, and $o(F)$ denotes a quantity q satisfying $q/F \to 0$ as $F \to 0$. Since the present system has no electron-electron interaction, the order estimate for the nonlinear part $\mathbf{j}'_{\mathrm{ind}}(t)$ in (2.24) holds also in the infinite volume limit [8, 9, 35] with the same form of the linear part of the induced current. But the nonlinear part $\mathbf{j}'_{\mathrm{ind}}(t)$ depends on the adiabatic parameters η and T. Therefore we cannot take the adiabatic limit $T \uparrow \infty$ and $\eta \downarrow 0$ for the nonlinear part $\mathbf{j}'_{\mathrm{ind}}(t)$ of the induced current.

Now we describe our main theorems. Let $\nu = N/M$ be the filling factor of the electrons for a finite volume, where N is the number of the electrons, and write $\omega_c = eB/m_e$ for the cyclotron frequency. Consider first the case with $\mathbf{A}_{\mathrm{P}} = 0$ in the infinite volume limit.

Theorem 2.1 *Assume that the filling factor ν satisfies $n - 1 < \nu \le n$ with a positive integer n. Then there exist positive constants, $B_0(n)$ and v_0, such that there appear localized states of the bulk order around the energy $\mathcal{E}_{n-1} = (n - 1/2)\hbar\omega_c$, i.e., the n-th Landau band center, for any magnetic field $B > B_0(n)$ and for any potential $V_0 \in C^2$ satisfying $\|V_0^+\|_\infty + \|V_0^-\|_\infty \le v_0$. Further, when the Fermi level lies in the localization regime, the conductances are quantized as*

$$\sigma_{xy} = -\frac{e^2}{h} \times \begin{cases} n & \text{for the upper localization regime,} \\ (n-1) & \text{for the lower localization regime,} \end{cases} \qquad \sigma_{yy} = 0, \tag{2.26}$$

and exhibit the plateaus for varying the filling factor. With probability one, there exist positive constants, $\mathcal{C}_j(\omega) < \infty$, $j = 1, 2, 3$, such that the corrections $\delta\sigma_{sy}(t)$ due to the initial adiabatic process satisfy

$$|\delta\sigma_{sy}(t)| \le [\mathcal{C}_1(\omega) + \mathcal{C}_2(\omega)T]e^{-\eta T} + \mathcal{C}_3(\omega)\eta^{1/13}, \tag{2.27}$$

and that the expectation $\mathbf{E}[\mathcal{C}_j]$ of the positive constants $\mathcal{C}_j(\omega)$ is finite for $j = 1, 2, 3$, i.e., $\mathbf{E}[\mathcal{C}_j] < \infty$. Here the constant $\mathcal{C}_j(\omega)$ itself without the expectation may depend on the random event ω of the random potential V_ω.

In the case with $\mathbf{A}_{\mathrm{P}} \ne 0$ in the infinite volume limit, we require a strong disorder because of a technical reason. (See Appendix D for details.) We take $u = \hbar\omega_c\hat{u}$ with a fixed, dimensionless function $\hat{u}$ for the random potential V_ω of (2.6). This potential behaves as $\|u\|_\infty \sim \text{Const.} \times B$ for a large B. For this random potential, we have:

Theorem 2.2 *Assume that the filling factor ν satisfies $n - 1 < \nu \le n$. Then there exist positive constants, $B_0(n), \alpha_0(n)$ and w_0, such that there appear localized states of the bulk order around the energy $\mathcal{E}_{n-1} = (n - 1/2)\hbar\omega_c$ for any magnetic field $B > B_0(n)$ and for any vector potential $\mathbf{A}_{\mathrm{P}}$ satisfying $\||\mathbf{A}_{\mathrm{P}}|\|_\infty \le \alpha_0(n)B^{1/2}$ and for the function $\hat{u}$ satisfying $\|\hat{u}\|_\infty \le w_0$. Further, when the Fermi level lies in the localization regime, the conductances, σ_{xy} and σ_{yy}, are quantized as in (2.26) and exhibit the plateaus for varying the filling factor. With probability one, the corrections $\delta\sigma_{sy}(t)$ due to the initial adiabatic process satisfy a bound having the same form as that of the bound (2.27).*

Remark

1. In the conditions in the first theorem, we can also take $u = \hbar\omega_c\hat{u}$ for the random potential with a small norm $\|\hat{u}\|_\infty$.
2. In the conditions in the second theorem, we can take V_0 which behaves as $\|V_0\|_\infty \sim \text{Const} \times B$ for a large B with a small positive constant, instead of a fixed potential V_0.
3. These two theorems do not necessarily state that both of the upper and lower band edges exhibit the localized states of the bulk order. Namely a localization region of the bulk order may appears only at one side for a single Landau band.
4. We do not require the assumption that the n-th Landau band is separated from the rest of the spectrum by two spectral gaps.
5. In the previous analyses [15–17, 25], they considered the Hamiltonian having the form with $\mathbf{A}_{\mathrm{P}} = 0$ and $V_0 = 0$. The analyses rely on the special properties of the unperturbed Hamiltonian. For example, they use the explicit forms of the integral kernel of the projections onto the Landau levels. The extension to the case with $\mathbf{A}_{\mathrm{P}} \ne 0$ and $V_0 \ne 0$ needs additional, non-trivial analyses for localization. In addition, we do not require a periodicity of the potentials $\mathbf{A}_{\mathrm{P}}$, V_0 with a finite period. Therefore the Hamiltonian H_ω does not need to be translation ergodic.
6. The widths of the plateaus can be estimated as we will show in Sect. 11 below. In particular, when $\mathbf{A}_{\mathrm{P}} = 0$ and $V_0 = 0$, the ratio of the localized states to the total number M of the states in the single Landau level tends to one as the strength B of the magnetic field goes to infinity. This implies that our estimate for the widths of the plateaus shows the optimal, expected value in this limit.

3 Spectral Gaps of the Hamiltonian H_0

In order to show the localization for the disordered-broadened Landau bands, we first need to check the condition for the appearance of the spectral gaps in the spectrum of the Hamiltonian H_0 of (2.2) with a generic, bounded potential V_0.

First we recall the simplest Landau Hamiltonian for a single electron only in the uniform magnetic field. The Hamiltonian is given by

$$H_{\mathrm{L}} = \frac{1}{2m_e}(\mathbf{p} + e\mathbf{A}_0)^2. \tag{3.1}$$

We assume that the electron is confined to the same finite rectangular box Λ^{sys} of (2.3) as the box for the Hamiltonian H_ω of (2.1), and impose the periodic boundary conditions (2.16) for the wavefunctions with the flux quantization condition $L_x L_y = 2\pi M \ell_B^2$. Then the energy eigenvalues $\mathcal{E}_n$ of H_{L} are given by[4]

$$\mathcal{E}_n := \left(n + \frac{1}{2}\right)\hbar\omega_c \quad \text{for } n = 0, 1, 2, \ldots. \tag{3.2}$$

The Hamiltonian H_0 of (2.2) on the finite box Λ^{sys} is written

$$\begin{aligned} H_0 &= \frac{1}{2m_e}(\mathbf{p} + e\mathbf{A}_0 + e\mathbf{A}_{\mathrm{P}})^2 + V_0 \\ &= H_{\mathrm{L}} + \frac{e}{2m_e}\mathbf{A}_{\mathrm{P}} \cdot (\mathbf{p} + e\mathbf{A}_0) + \frac{e}{2m_e}(\mathbf{p} + e\mathbf{A}_0) \cdot \mathbf{A}_{\mathrm{P}} + \frac{e^2}{2m_e}|\mathbf{A}_{\mathrm{P}}|^2 + V_0. \end{aligned} \tag{3.3}$$

Using the Schwarz inequality, one has

$$|(\psi, \mathbf{A}_{\mathrm{P}} \cdot (\mathbf{p} + e\mathbf{A}_0)\psi)| \le \||\mathbf{A}_{\mathrm{P}}|\|_\infty \sqrt{(\psi, (\mathbf{p} + e\mathbf{A}_0)^2\psi)} \tag{3.4}$$

for the normalized vector ψ in the domain of the Hamiltonian. From this inequality, the energy expectation can be evaluated as

$$(\psi, H_0\psi) \le (\psi, H_{\mathrm{L}}\psi) + \frac{\sqrt{2}e}{\sqrt{m_e}}\||\mathbf{A}_{\mathrm{P}}|\|_\infty \sqrt{(\psi, H_{\mathrm{L}}\psi)} + \frac{e^2}{2m_e}\||\mathbf{A}_{\mathrm{P}}|\|_\infty^2 + \|V_0^+\|_\infty \tag{3.5}$$

and

$$(\psi, H_0\psi) \ge (\psi, H_{\mathrm{L}}\psi) - \frac{\sqrt{2}e}{\sqrt{m_e}}\||\mathbf{A}_{\mathrm{P}}|\|_\infty \sqrt{(\psi, H_{\mathrm{L}}\psi)} - \|V_0^-\|_\infty, \tag{3.6}$$

where $V_0^\pm = \max\{\pm V_0, 0\}$. Let us denote by $\mathcal{E}_{n,+}^{\mathrm{edge}}$ and $\mathcal{E}_{n,-}^{\mathrm{edge}}$, respectively, the upper and lower edges of the $n+1$-th Landau band which is broadened by the potentials V_0 and $\mathbf{A}_{\mathrm{P}}$. From the standard argument about the min-max principle,[5] one has

$$\mathcal{E}_{n,+}^{\mathrm{edge}} \le \mathcal{E}_n + \frac{\sqrt{2}e}{\sqrt{m_e}}\||\mathbf{A}_{\mathrm{P}}|\|_\infty \sqrt{\mathcal{E}_n} + \frac{e^2}{2m_e}\||\mathbf{A}_{\mathrm{P}}|\|_\infty^2 + \|V_0^+\|_\infty \tag{3.7}$$

[4] See, for example, Refs. [6, 36].

[5] See, for example, Sect. XIII.1 of the book [27] by M. Reed and B. Simon.

for $n = 0, 1, 2, \ldots$. For the lower edge, we assume

$$\frac{e}{\sqrt{2m_e}} \| |\mathbf{A}_{\mathrm{P}}| \|_\infty \le \sqrt{\frac{1}{2}\hbar\omega_c}. \tag{3.8}$$

Then the right-hand side of the bound (3.6) is a strictly monotone increasing function of the expectation $(\psi, H_{\mathrm{L}}\psi)$. Therefore, the same argument yields

$$\mathcal{E}_{n,-}^{\mathrm{edge}} \ge \mathcal{E}_n - \frac{\sqrt{2}e}{\sqrt{m_e}} \| |\mathbf{A}_{\mathrm{P}}| \|_\infty \sqrt{\mathcal{E}_n} - \| V_0^- \|_\infty \tag{3.9}$$

for $n = 0, 1, 2, \ldots$. If this right-hand side with the index $n+1$ is strictly larger than the right-hand side of (3.7) with the index n, then there exists a spectral gap above the Landau band with the index n, i.e., $\mathcal{E}_{n+1,-}^{\mathrm{edge}} > \mathcal{E}_{n,+}^{\mathrm{edge}}$. This gap condition can be written as

$$\hbar\omega_c > \frac{\sqrt{2}e}{\sqrt{m_e}} \| |\mathbf{A}_{\mathrm{P}}| \|_\infty (\sqrt{\mathcal{E}_{n+1}} + \sqrt{\mathcal{E}_n}) + \frac{e^2}{2m_e} \| |\mathbf{A}_{\mathrm{P}}| \|_\infty^2 + \| V_0^+ \|_\infty + \| V_0^- \|_\infty. \tag{3.10}$$

Clearly this is stronger than the condition (3.8) for the vector potential $\mathbf{A}_{\mathrm{P}}$. Therefore we have no need to take into account the condition (3.8).

4 Site Percolation on the Triangular Lattice

The classical motion of the electron is forbidden in the regions that the strength of the potential is smaller than the deviation of the energy of the electron from the Landau energies $\mathcal{E}_n$ of (3.2). In those regions, the Green function of the electron decays exponentially. In order to get the decay bound for the Green function, we study the distribution of those classically forbidden regions. We reformulate this problem as a site percolation problem on the triangular lattice. The idea of using percolation is due to Combes and Hislop [15] or Wang [16]. But both of their random potentials are different from the present potential which we require for estimating the number of the localized states.

We begin with setting up site percolation on the triangular lattice $\mathbf{L}^2$ for the present random potential. We say that the site $\mathbf{z} \in \mathbf{L}^2$ is occupied if $\lambda_{\mathbf{z}}(\omega) \in (-\lambda_-, \lambda_+)$. The probability p that a site $\mathbf{z}$ is occupied is given by

$$p = \int_{-\lambda_-}^{\lambda_+} g(\lambda) d\lambda. \tag{4.1}$$

The assumption (2.7) implies $p > p_c = 1/2$. Here p_c is the critical probability which equals $1/2$ for the present site percolation on the triangular lattice [37, 38]. A path of $\mathbf{L}^2$ is a sequence $\mathbf{z}_0, \mathbf{z}_1, \ldots, \mathbf{z}_n$ of sites $\mathbf{z}_j$ such that all of the adjacent two site $\mathbf{z}_j, \mathbf{z}_{j+1}$ are corresponding to a side of a unit triangle. If $\mathbf{z}_0 = \mathbf{z}_n$, then we say that the path is closed, and we call a closed path a circuit. If all of the site $\mathbf{z}_j$ of the path are occupied, then we say that the path is occupied. Similarly we define an unoccupied path, an occupied circuit, etc. We denote by $P_p(A)$ the probability that an event A occurs.

Let $\Pi_{\ell,\ell'}$ be a parallelogram with the lengths ℓa, $\ell' a$ of the sides in the triangular lattice. See Fig. 1. More precisely, it is given by

$$\Pi_{\ell,\ell'} := \left\{ m\mathbf{a}_1 + n\mathbf{a}_2 \,\middle|\, |m| \le \frac{\ell}{2},\ |n| \le \frac{\ell'}{2},\ (m,n) \in \mathbf{Z}^2 \right\}. \tag{4.2}$$

Here we take ℓ, ℓ' even integers for simplicity. Consider the event $\bar{A}_{\ell,\ell'}$ that there exists a unoccupied path in the parallelogram $\Pi_{\ell,\ell'}$ joining a site on the lower side with the length ℓ to a site on the upper side. Since the connectivity between two sites with an unoccupied path decays exponentially for $p > p_c$, the probability $P_p(\bar{A}_{\ell,\ell'})$ that the event $\bar{A}_{\ell,\ell'}$ occurs is bounded as

$$P_p(\bar{A}_{\ell,\ell'}) \le \mathrm{Const} \times \ell \exp[-m_p \ell'] \quad \text{for } p > p_c = 1/2, \tag{4.3}$$

where m_p is a positive function of p. Let $A_{\ell',\ell}$ be the event that there exists an occupied path in the parallelogram $\Pi_{\ell,\ell'}$ joining a site on the left side with the length ℓ' to a site on the right side. Then this event $A_{\ell',\ell}$ is the complementary event of $\bar{A}_{\ell,\ell'}$ because of the structure of the triangular lattice. Immediately,

$$P_p(A_{\ell',\ell}) + P_p(\bar{A}_{\ell,\ell'}) = 1. \tag{4.4}$$

Combining this with the above inequality, one has

$$P_p(A_{\ell',\ell}) \ge 1 - \mathrm{Const} \times \ell \exp[-m_p \ell'] \quad \text{for } p > p_c = 1/2. \tag{4.5}$$

Consider a parallelogram-shaped region consisting of hexagons such that

$$\Lambda^{\mathrm{para}}_{\ell+1,\ell'+1}(\mathbf{z}_0) := \bigcup_{\mathbf{z} \in \Pi_{\ell,\ell'}} h_{\mathbf{z}+\mathbf{z}_0}, \tag{4.6}$$

where $h_{\mathbf{z}}$ is the region of the face of the hexagon with the center $\mathbf{z}$, including the six sides of the hexagon, and $\mathbf{z}_0$ is the center of the region $\Lambda^{\mathrm{para}}_{\ell+1,\ell'+1}(\mathbf{z}_0)$. The region has the jagged boundary as in Fig. 1. Moreover we define an annular region as

$$\Lambda^{\mathrm{annu}}_{3\ell,3\ell'}(\mathbf{z}_0) := \Lambda^{\mathrm{para}}_{3\ell,3\ell'}(\mathbf{z}_0) \backslash \Lambda^{\mathrm{para}}_{\ell,\ell'}(\mathbf{z}_0), \tag{4.7}$$

where both ℓ and ℓ' take an odd integer.

Let us consider the event $D_{\ell,\ell'}$ in $\Lambda^{\mathrm{annu}}_{3\ell,3\ell'}(\mathbf{z}_0) \cap \mathbf{L}^2$ that there exists an occupied circuit $\mathcal{C}$ encircling the inside region $\Lambda^{\mathrm{para}}_{\ell,\ell'}(\mathbf{z}_0)$. Then, from the inequality (4.5) and FKG inequality,[6] the probability $P_p(D_{\ell,\ell'})$ for this event satisfies

$$\begin{aligned} P_p(D_{\ell,\ell'}) &\ge [P_p(A_{\ell',3\ell})]^2 [P_p(A_{\ell,3\ell'})]^2 \\ &\ge 1 - \mathrm{Const} \times \{\ell \exp[-m_p \ell'] + \ell' \exp[-m_p \ell]\} \\ &\text{for } p > p_c = 1/2. \end{aligned} \tag{4.8}$$

Therefore the event of an occupied circuit occurs with the probability nearly equal to one for large ℓ, ℓ'.

We denote by $b_{j,j+1}$ the side $\overline{\mathbf{z}_j \mathbf{z}_{j+1}}$ of a unit triangle, i.e.,

$$b_{j,j+1} := \{\mathbf{r} = \lambda \mathbf{z}_j + (1-\lambda)\mathbf{z}_{j+1} | \lambda \in [0,1]\}. \tag{4.9}$$

We define the region $\mathcal{R}_{j,j+1}$ including the side $b_{j,j+1}$ as

$$\mathcal{R}_{j,j+1} := \{\mathbf{r} | \mathrm{dist}(\mathbf{r}, b_{j,j+1}) \le r_1\} \quad \text{with } r_1 := \frac{\sqrt{3}}{2} a - r_u, \tag{4.10}$$

[6] See, for example, the book [38].

where r_u is given in the condition (2.9) for the bump u. Further we define the ribbon region $\mathcal{R}_{\mathcal{C}}$ associated a circuit $\mathcal{C}$ by

$$\mathcal{R}_{\mathcal{C}} := \bigcup_{b_{j,j+1} \in \mathcal{C}} \mathcal{R}_{j,j+1}. \tag{4.11}$$

Clearly r_1 is strictly positive from the condition $r_u \in (\sqrt{3}a/3, \sqrt{3}a/2)$ of (2.9), and the ribbon region $\mathcal{R}_{\mathcal{C}}$ has a nonzero width $2r_1$.

Proposition 4.1 *There appears an occupied circuit $\mathcal{C}$ in the annular region $\Lambda^{\text{annu}}_{3\ell,3\ell'}(\mathbf{z}_0)$ with a probability larger than*

$$P^{\text{perc}} := 1 - C^{\text{perc}}\{\ell \exp[-m_p \ell'] + \ell' \exp[-m_p \ell]\}, \tag{4.12}$$

where C^{perc} is the positive constant in the right-hand side of the above bound (4.8), and $p > p_c = 1/2$ is given by (4.1). Further the following hold for the ribbon region $\mathcal{R}_{\mathcal{C}}$ associated with the occupied circuit $\mathcal{C}$:

$$\operatorname{dist}(\mathcal{R}_{\mathcal{C}}, \partial \Lambda^{\text{annu}}_{3\ell,3\ell'}(\mathbf{z}_0)) = r_u - \frac{\sqrt{3}}{3}a =: r_2 > 0 \tag{4.13}$$

and

$$-\lambda_- u_1 \le V_\omega(\mathbf{r}) \le \lambda_+ u_1 \quad \textit{for } \mathbf{r} \in \mathcal{R}_{\mathcal{C}}. \tag{4.14}$$

Here $\partial \Lambda^{\text{annu}}_{3\ell,3\ell'}(\mathbf{z}_0)$ is the boundary of the annular region $\Lambda^{\text{annu}}_{3\ell,3\ell'}(\mathbf{z}_0)$, and $u_1 := 2\|u\|_\infty$.

Proof The lower bound (4.12) of the probability is nothing but the right-hand side of (4.8). The positivity (4.13) of r_2 follows from the condition (2.9) of the bump u of the random potential V_ω, and the bound (4.14) follows from the condition (2.9) and the definition (2.6) of the random potential V_ω. □

5 Initial Decay Estimate for the Resolvent

Now let us estimate the decay of the resolvent (Green function) for a finite parallelogram-shaped region. The resulting decay bound in Proposition 5.2 below will become the initial data for the multi-scale analysis to obtain the decay bounds for the resolvent in larger scales in the next section. However, by using the multi-scale analysis, we cannot get a similar decay bound for the resolvent for two arbitrary points in the infinite-volume limit. On the other hand, the fractional moment method leads us to a decay bound for a fractional moment of the resolvent in the infinite-volume limit. Actually, as we will see in Sect. 7, the initial decay estimate of this section yields such a decay bound. But the resolvent itself without taking a fractional moment cannot be evaluated by the method [32]. Due to technical reason related to these observations, we need both multi-scale analysis and fractional moment analysis, in order to prove the existence of the conductance plateaus with a bulk order width.

Although the method in this section is basically the same as in the previous papers [15, 16] as mentioned in the preceding section, we need more detailed analysis about the magnetic field dependence of the decay bounds, in order to estimate the number of the localized states which yield the Hall conductance plateau with a bulk order width.

Fix the random variables $\lambda_{\mathbf{z}+\mathbf{z}_0}$ with $\mathbf{z} \in \mathbf{L}^2 \backslash \Pi_{3\ell-1,3\ell'-1}$. Here ℓ and ℓ' are odd integers larger than 1. Consider the parallelogram-shaped region $\Lambda^{\text{para}}_{3\ell,3\ell'}(\mathbf{z}_0)$ centered at $\mathbf{z}_0$, and assume $\Lambda_{3\ell,3\ell'}^{\mathrm{para}}(\mathbf{z}_0) \subset \Lambda^{\mathrm{sys}}$ with a sufficiently large box Λ^{sys} of (2.3). We write $\Lambda_{3\ell,3\ell'} = \Lambda_{3\ell,3\ell'}^{\mathrm{para}}(\mathbf{z}_0)$ for short. Further we consider the Hamiltonian H_ω restricting to the region $\Lambda_{3\ell,3\ell'}$ with the Dirichlet boundary conditions. The Hamiltonian is written as

$$H_{\Lambda_{3\ell,3\ell'}} = \frac{1}{2m_e}(\mathbf{p}+e\mathbf{A})^2 + V_0|_{\Lambda_{3\ell,3\ell'}} + \hat{V}_{\omega,3\ell,3\ell'} + \delta V_{\omega,3\ell,3\ell'}, \tag{5.1}$$

where we have decomposed the random potential V_ω into two parts,

$$\hat{V}_{\omega,3\ell,3\ell'}(\mathbf{r}) = \sum_{\mathbf{z}\in\Pi_{3\ell-1,3\ell'-1}} \lambda_{\mathbf{z}+\mathbf{z}_0}(\omega)u(\mathbf{r}-\mathbf{z}_0-\mathbf{z}) \tag{5.2}$$

and $\delta V_{\omega,3\ell,3\ell'} = V_\omega|_{\Lambda_{3\ell,3\ell'}} - \hat{V}_{\omega,3\ell,3\ell'}$. Clearly, the first part $\hat{V}_{\omega,3\ell,3\ell'}$ of the random potential is determined by only the random variables $\lambda_{\mathbf{z}+\mathbf{z}_0}(\omega)$ at the sites $\mathbf{z}+\mathbf{z}_0$ lying in the parallelogram-shaped region $\Lambda_{3\ell,3\ell'}^{\mathrm{para}}(\mathbf{z}_0)$, and so it is independent of the outside random variables. Further we have

$$\sum_{\mathbf{z}\in\Pi_{3\ell-1,3\ell'-1}} u(\mathbf{r}-\mathbf{z}_0-\mathbf{z}) \ge u_0 \quad \text{for any } \mathbf{r}\in\Lambda_{3\ell,3\ell'} \tag{5.3}$$

from the assumption (2.10). This condition will be useful to obtain the Wegner estimate [39] for the density of the states. (See Appendix A for the Wegner estimate.)

On the other hand, the random potential $\delta V_{\omega,3\ell,3\ell'}$ which is supported by only the region near the boundary of $\Lambda_{3\ell,3\ell'}$, depends on the outside random variables. Following [15], we absorb this term into the operator $W(\chi)$ of (5.7) below which will appear in the geometric resolvent equation. (See Appendix F for details.) Thus we consider the Hamiltonian,

$$\hat{H}_{\Lambda_{3\ell,3\ell'}} = \frac{1}{2m_e}(\mathbf{p}+e\mathbf{A})^2 + V_0|_{\Lambda_{3\ell,3\ell'}} + \hat{V}_{\omega,3\ell,3\ell'}, \tag{5.4}$$

without the potential $\delta V_{\omega,3\ell,3\ell'}$, instead of the Hamiltonian $H_{\Lambda_{3\ell,3\ell'}}$ of (5.1).

Assume that the energy $E\in\mathbf{R}$ satisfies the condition,

$$\mathcal{E}_{n,+}^{\mathrm{edge}} + \lambda_+ u_1 < E < \mathcal{E}_{n+1,-}^{\mathrm{edge}} - \lambda_- u_1 \quad \text{with } u_1 = 2\|u\|_\infty. \tag{5.5}$$

We write the resolvent as $R_{3\ell,3\ell'} = R_{3\ell,3\ell'}(E+i\varepsilon) = (\hat{H}_{\Lambda_{3\ell,3\ell'}} - E - i\varepsilon)^{-1}$ with $\varepsilon\in\mathbf{R}$. For $\delta\in(0,r_2)$, consider the region, $\Lambda_{3\ell,3\ell'}^{\delta} := \{\mathbf{r}\in\Lambda_{3\ell,3\ell'} | \mathrm{dist}(\mathbf{r},\partial\Lambda_{3\ell,3\ell'}) > \delta\}$, where r_2 is given by (4.13), and $\partial\Lambda_{3\ell,3\ell'}$ is the boundary of the region $\Lambda_{3\ell,3\ell'}$. Let $\chi_{3\ell,3\ell'}^{\delta}$ be a C^2, positive cut-off function which satisfies

$$\chi_{3\ell,3\ell'}^{\delta}|_{\Lambda_{3\ell,3\ell'}^{\delta}} = 1 \quad \text{and} \quad \mathrm{supp}\,|\nabla\chi_{3\ell,3\ell'}^{\delta}| \subset \Lambda_{3\ell,3\ell'}\backslash\Lambda_{3\ell,3\ell'}^{\delta}. \tag{5.6}$$

We denote by $\chi_{\ell,\ell'}$ the characteristic function of the region $\Lambda_{\ell,\ell'}^{\mathrm{para}}(\mathbf{z}_0)$.

The purpose of this section is to estimate the decay of $W(\chi_{3\ell,3\ell'}^{\delta})R_{3\ell,3\ell'}\chi_{\ell,\ell'}$, where

$$W(\chi) = [(\mathbf{p}+e\mathbf{A})^2/(2m_e), \chi] \tag{5.7}$$

for a C^2 function χ. Note that

$$\|W(\chi^{\delta}_{3\ell,3\ell'})R_{3\ell,3\ell'}\chi_{\ell,\ell'}\| \le \frac{\hbar^2}{2m_e}\|(\Delta\chi^{\delta}_{3\ell,3\ell'})R_{3\ell,3\ell'}\chi_{\ell,\ell'}\| + \frac{\hbar}{m_e}\sum_{i=x,y}\|(\partial_i\chi^{\delta}_{3\ell,3\ell'})(p_i+eA_i)R_{3\ell,3\ell'}\chi_{\ell,\ell'}\|, \tag{5.8}$$

where we have written $\nabla = (\partial_x, \partial_y)$, and used

$$W(\chi) = -\frac{i\hbar}{m_e}(\mathbf{p}+e\mathbf{A})\cdot\nabla\chi + \frac{\hbar^2}{2m_e}\Delta\chi = -\frac{i\hbar}{m_e}\nabla\chi\cdot(\mathbf{p}+e\mathbf{A}) - \frac{\hbar^2}{2m_e}\Delta\chi. \tag{5.9}$$

We write $\mathcal{R}$ for the ribbon region $\mathcal{R}_{\mathcal{C}}$ in Proposition 4.1 for short. Let ϵ be a small positive number, and let $\mathcal{C}_\epsilon := \{\mathbf{r}\in\mathcal{R} \mid \mathrm{dist}(\mathbf{r},\mathcal{C}) < \epsilon/2\}$, so that the region $\mathcal{C}_\epsilon$ has the width ϵ, where $\mathcal{C}$ is the occupied, closed path in Proposition 4.1. Let χ_1 be a C^2, positive cut-off function satisfying $\chi_1|_{\Lambda_{\ell,\ell'}} = 1$ and $\mathrm{supp}\,|\nabla\chi_1| \subset \mathcal{C}_\epsilon$, where we have written $\Lambda_{\ell,\ell'}$ for $\Lambda^{\mathrm{para}}_{\ell,\ell'}(\mathbf{z}_0)$ for short. Since we can take χ_1 to satisfy $(\partial_i\chi^{\delta}_{3\ell,3\ell'})\chi_1 = 0$ from the definition of $\chi^{\delta}_{3\ell,3\ell'}$, one has

$$(\partial_i\chi^{\delta}_{3\ell,3\ell'})(p_i+eA_i)R_{3\ell,3\ell'}\chi_{\ell,\ell'} = D_iR_{3\ell,3\ell'}\chi_1\chi_{\ell,\ell'} = -D_iR_{3\ell,3\ell'}W(\chi_1)R_{3\ell,3\ell'}\chi_{\ell,\ell'}, \tag{5.10}$$

where we have written $D_i = (\partial_i\chi^{\delta}_{3\ell,3\ell'})(p_i+eA_i)$. In order to estimate this right-hand side, we define the ϵ border of the ribbon region $\mathcal{R}$ by $\mathcal{R}_\epsilon := \{\mathbf{r}\in\mathcal{R}|\mathrm{dist}(\mathbf{r},\partial\mathcal{R}) < \epsilon\}$, and define $r_3 := \mathrm{dist}(\mathcal{R}_\epsilon, \mathcal{C}_\epsilon) > 0$. We choose a small parameter ϵ so that the distance r_3 becomes strictly positive. Further we introduce two C^2, positive cut-off functions, $\tilde{\chi}^{\epsilon/2}_{\mathcal{R}}$ and $\chi^{\epsilon}_{\mathcal{R}}$, which satisfy the following conditions:

$$\tilde{\chi}^{\epsilon/2}_{\mathcal{R}}|_{\mathcal{R}\setminus\mathcal{R}_{\epsilon/2}} = 1, \qquad \mathrm{supp}\,|\nabla\tilde{\chi}^{\epsilon/2}_{\mathcal{R}}| \subset \mathcal{R}_{\epsilon/2}, \tag{5.11}$$

and

$$\chi^{\epsilon}_{\mathcal{R}}|_{\mathcal{R}\setminus\mathcal{R}_\epsilon} = 1, \qquad \chi^{\epsilon}_{\mathcal{R}}|_{\mathcal{R}_{\epsilon/2}} = 0, \quad \text{and} \quad \mathrm{supp}\,|\nabla\chi^{\epsilon}_{\mathcal{R}}| \subset \mathcal{R}_\epsilon\setminus\mathcal{R}_{\epsilon/2}. \tag{5.12}$$

Consider the Hamiltonian,

$$H_{\mathcal{R}} := \frac{1}{2m_e}(\mathbf{p}+e\mathbf{A})^2 + V_{\mathcal{R}}, \tag{5.13}$$

on the finite rectangular box Λ^{sys} of (2.3), where we impose the periodic boundary conditions (2.16) with the flux quantization condition $L_xL_y = 2\pi M\ell_B^2$, and the potential is given by $V_{\mathcal{R}} = \tilde{\chi}^{\epsilon/2}_{\mathcal{R}}(V_0+V_\omega)$. Then one has the geometric resolvent equation, $R_{3\ell,3\ell'}\chi^{\epsilon}_{\mathcal{R}} = \chi^{\epsilon}_{\mathcal{R}}R_{\mathcal{R}} - R_{3\ell,3\ell'}W(\chi^{\epsilon}_{\mathcal{R}})R_{\mathcal{R}}$ with $R_{\mathcal{R}} := (H_{\mathcal{R}} - E - i\epsilon)^{-1}$, where we have used $\tilde{\chi}^{\epsilon/2}_{\mathcal{R}}\chi^{\epsilon}_{\mathcal{R}} = \chi^{\epsilon}_{\mathcal{R}}$ which is easily obtained from the definitions. Using this equation and $D_i\chi^{\epsilon}_{\mathcal{R}} = 0$, the right-hand side of (5.10) is written as

$$-D_iR_{3\ell,3\ell'}W(\chi_1)R_{3\ell,3\ell'}\chi_{\ell,\ell'} = -D_iR_{3\ell,3\ell'}\chi^{\epsilon}_{\mathcal{R}}W(\chi_1)R_{3\ell,3\ell'}\chi_{\ell,\ell'} = D_iR_{3\ell,3\ell'}W(\chi^{\epsilon}_{\mathcal{R}})R_{\mathcal{R}}W(\chi_1)R_{3\ell,3\ell'}\chi_{\ell,\ell'}. \tag{5.14}$$

Consequently, one obtains

$$D_iR_{3\ell,3\ell'}\chi_{\ell,\ell'} = D_iR_{3\ell,3\ell'}W(\chi^{\epsilon}_{\mathcal{R}})R_{\mathcal{R}}W(\chi_1)R_{3\ell,3\ell'}\chi_{\ell,\ell'}. \tag{5.15}$$

In the same way,

$$(\Delta\chi^{\delta}_{3\ell,3\ell'})R_{3\ell,3\ell'}\chi_{\ell,\ell'} = (\Delta\chi^{\delta}_{3\ell,3\ell'})R_{3\ell,3\ell'}W(\chi^{\epsilon}_{\mathcal{R}})R_{\mathcal{R}}W(\chi_1)R_{3\ell,3\ell'}\chi_{\ell,\ell'}. \tag{5.16}$$

If the random potential V_ω satisfies the condition (4.14), then the energy E satisfying the condition (5.5) is in the spectral gap of the Hamiltonian $H_{\mathcal{R}}$. Therefore we can apply the Combes-Thomas method [40] to evaluate decay of the resolvent $R_{\mathcal{R}}$ in the right-hand sides of (5.15) and (5.16). We write $\mathcal{A} = \mathrm{supp}\,|\nabla\chi_{\mathcal{R}}^{\epsilon}|$ and $\mathcal{B} = \mathcal{C}_\epsilon$, and denote by $\chi_{\mathcal{A}}$ and $\chi_{\mathcal{B}}$ the characteristic function of $\mathcal{A}$ and $\mathcal{B}$, respectively. The resolvent $R_{\mathcal{R}}$ decays as

$$\|\chi_{\mathcal{A}} R_{\mathcal{R}} \chi_{\mathcal{B}}\| \le C_1^{(n)} e^{-\beta r_3} \tag{5.17}$$

with a probability larger than P^{perc} of (4.12) in Proposition 4.1, where $C_1^{(n)}$ and β are positive constants. The derivation of this decay bound and the explicit parameter dependence of $C_1^{(n)}$ and of β for the present model are given in Appendix D. In the application of the decay bound, we choose the cut-off function $\chi_{\mathcal{R}}^{\epsilon}$ so that the region $\mathrm{supp}\,|\nabla\chi_{\mathcal{R}}^{\epsilon}|$ has a smooth boundary.

Lemma 5.1 *Let $R = (H_\omega - E - i\varepsilon)^{-1}$ with generic, bounded potentials V_0, V_ω and $E, \varepsilon \in \mathbf{R}$ satisfying $E \notin \sigma(H_\omega)$ or $\varepsilon \neq 0$, and let $\boldsymbol{\alpha} = (\alpha_x, \alpha_y)$ be a vector-valued C^1 function. Then*

$$\|\boldsymbol{\alpha}\cdot(\mathbf{p}+e\mathbf{A})R\| \le 2\sqrt{2m_e}\|R\|^{1/2}(1+f_{E,R})^{1/2}\max_{i=x,y}\{\|\alpha_i\|_\infty\}, \tag{5.18}$$

$$\|(p_i+eA_i)R(\mathbf{p}+e\mathbf{A})\cdot\boldsymbol{\alpha}\| \le 2m_e\||\boldsymbol{\alpha}|\|_\infty(1+f_{E,R}) \tag{5.19}$$

and

$$\|R(\mathbf{p}+e\mathbf{A})\cdot\boldsymbol{\alpha}\| \le \sqrt{2m_e}\||\boldsymbol{\alpha}|\|_\infty\|R\|^{1/2}(1+f_{E,R})^{1/2}, \tag{5.20}$$

where we have written

$$f_{E,R} = [|E| + \|(V_0^- + V_\omega^-)\|_\infty]\|R\|. \tag{5.21}$$

The proof is given in Appendix E. From (5.9) and these bounds of Lemma 5.1, one has

$$\|W(\chi_1)R_{3\ell,3\ell'}\| \le f_1(|E|, \|R_{3\ell,3\ell'}\|), \tag{5.22}$$

$$\|R_{3\ell,3\ell'}W(\chi_{\mathcal{R}}^{\epsilon})\| \le f_2(|E|, \|R_{3\ell,3\ell'}\|) \tag{5.23}$$

and

$$\|(p_i+eA_i)R_{3\ell,3\ell'}W(\chi_{\mathcal{R}}^{\epsilon})\| \le f_3(|E|, \|R_{3\ell,3\ell'}\|), \tag{5.24}$$

for the operators in the right-hand sides of (5.15) and (5.16), where the functions, f_1, f_2 and f_3, are given by

$$\begin{aligned} f_1(|E|, \|R\|) = {} & \frac{\hbar^2}{2m_e}\|\Delta\chi_1\|_\infty\|R\| \\ & + 2\hbar\sqrt{\frac{2}{m_e}}\{\|R\| + [|E| + \|(V_0^- + V_\omega^-)\|_\infty]\|R\|^2\}^{1/2} \\ & \times \max_{i=x,y}\{\|\partial_i\chi_1\|_\infty\}, \end{aligned} \tag{5.25}$$

$$f_2(|E|, \|R\|) = \frac{\hbar^2}{2m_e}\|\Delta\chi_{\mathcal{R}}^{\epsilon}\|_\infty\|R\| + \hbar\sqrt{\frac{2}{m_e}}\{\|R\| + [|E| + \|(V_0^- + V_\omega^-)\|_\infty]\|R\|^2\}^{1/2}\||\nabla\chi_{\mathcal{R}}^{\epsilon}|\|_\infty \tag{5.26}$$

and

$$f_3(|E|, \|R\|) = \frac{\hbar^2}{\sqrt{2m_e}}\|\Delta\chi_{\mathcal{R}}^{\epsilon}\|_\infty\{\|R\| + [|E| + \|(V_0^- + V_\omega^-)\|_\infty]\|R\|^2\}^{1/2} + 2\hbar\||\nabla\chi_{\mathcal{R}}^{\epsilon}|\|_\infty\{1 + [|E| + \|(V_0^- + V_\omega^-)\|_\infty]\|R\|\}. \tag{5.27}$$

The norm $\|R_{3\ell,3\ell'}\|$ of the resolvent in these upper bounds can be evaluated by using the Wegner estimate. See Appendix A for details. From the resulting Theorem A.2, we have that, for any $\delta E > 0$,

$$\|R_{3\ell,3\ell'}\| \le (\delta E)^{-1} \tag{5.28}$$

with a probability larger than

$$1 - C_{\mathrm{W}}K_3\|g\|_\infty\delta E|\Lambda_{3\ell,3\ell'}^{\mathrm{para}}(\mathbf{z}_0)|, \tag{5.29}$$

where C_{W} is a positive constant, and the positive constant K_3 is given by (A.39) in the theorem.

Proposition 5.2 *For any E satisfying the gap condition* (5.5), *and for any $\delta E > 0$, the following bound is valid*:

$$\begin{aligned}\sup_{\varepsilon\neq 0}&\|W(\chi_{3\ell,3\ell'}^{\delta})R_{3\ell,3\ell'}(E+i\varepsilon)\chi_{\ell,\ell'}\| \\ &\le C_1^{(n)}e^{-\beta r_3}f_1(|E|,(\delta E)^{-1})\left[\frac{\hbar^2}{2m_e}\|\Delta\chi_{3\ell,3\ell'}^{\delta}\|_\infty f_2(|E|,(\delta E)^{-1})\right. \\ &\quad\left. + \frac{2\hbar}{m_e}\max_{i=x,y}\|\partial_i\chi_{3\ell,3\ell'}^{\delta}\|_\infty f_3(|E|,(\delta E)^{-1})\right]\end{aligned} \tag{5.30}$$

with probability at least

$$P^{\mathrm{ini}} := 1 - \{C^{\mathrm{perc}}\{\ell\exp[-m_p\ell'] + \ell'\exp[-m_p\ell]\} + C_{\mathrm{W}}K_3\|g\|_\infty\delta E|\Lambda_{3\ell,3\ell'}^{\mathrm{para}}(\mathbf{z}_0)|\}. \tag{5.31}$$

Proof Combining (5.15), (5.16), (5.17), (5.22), (5.23), (5.24) and (5.28), the right-hand side of (5.8) is estimated. Using (4.12), (5.29) and the inequality $\mathrm{Prob}(A\cap B) \ge \mathrm{Prob}(A) + \mathrm{Prob}(B) - 1$, the probability is estimated. □

Since we can take the ribbon region $\mathcal{R}$ satisfying $\mathrm{supp}\,\delta V_{\omega,3\ell,3\ell'} \cap \mathcal{R} \subset \partial\mathcal{R}$, we can obtain a similar bound for $\|\delta V_{\omega,3\ell,3\ell'}R_{3\ell,3\ell'}(E+i\varepsilon)\chi_{\ell,\ell'}\|$ to (5.30). Fix the ratio ℓ'/ℓ. For simplicity we take $\ell' = \ell$. Fix $\xi > 4$. We choose $\ell = \ell_0$ to satisfy

$$2C^{\mathrm{perc}}\ell_0\exp[-m_p\ell_0] \le \ell_0^{-\xi}/2, \tag{5.32}$$

and choose δE in (5.28) so that

$$C_{\mathrm{W}}K_3\|g\|_\infty\delta E|\Lambda_{3\ell_0,3\ell_0}^{\mathrm{para}}(\mathbf{z}_0)| = \ell_0^{-\xi}/2. \tag{5.33}$$

Clearly these implies $P^{\mathrm{ini}} \ge 1 - \ell_0^{-\xi}$. Therefore, if we can find a large β in (5.30) so that the right-hand side of (5.30) with $\ell = \ell' = \ell_0$ becomes small, then we have

$$\mathrm{Prob}\Big[\sup_{\varepsilon\neq 0} \|\tilde{W}^{\delta}_{3\ell_0} R_{3\ell_0,3\ell_0}(E+i\varepsilon)\chi_{\ell_0,\ell_0}\| \le e^{-\gamma_0\ell_0}\Big] \ge 1 - \ell_0^{-\xi} \tag{5.34}$$

with some $\gamma_0 > 0$, where $\tilde{W}^{\delta}_{3\ell} := W(\chi^{\delta}_{3\ell,3\ell}) - \delta V_{\omega,3\ell,3\ell}\chi^{\delta}_{3\ell,3\ell}$.

For the norm $\|\chi_{\ell,\ell'} R_{3\ell,3\ell'}(E+i\varepsilon)W(\chi^{\delta}_{3\ell,3\ell'})\|$, one can obtain a similar bound to (5.30) in the same way. As a result, we have

$$\mathrm{Prob}\Big[\sup_{\varepsilon\neq 0} \|\chi_{\ell_0,\ell_0} R_{3\ell_0,3\ell_0}(E+i\varepsilon)(\tilde{W}^{\delta}_{3\ell_0})^*\| \le e^{-\gamma_0\ell_0}\Big] \ge 1 - \ell_0^{-\xi} \tag{5.35}$$

under the same assumption.

Let us study the condition which realizes such a large β. Consider first the case with $\mathbf{A}_{\mathrm{P}} = 0$, and fix the size $\ell = \ell_0$ of the box to satisfy the condition (5.32). Further, fix $\Delta\mathcal{E} > 0$ and $\hat{\delta}_{\pm} > 0$. Assume that the energy $E \in \mathbf{R}$ satisfies

$$\mathcal{E}_{n-1} + \|V_0^+\|_\infty + \lambda_+ u_1 + \hat{\delta}_-\hbar\omega_c \le E \le \mathcal{E}_n - \|V_0^-\|_\infty - \lambda_- u_1 - \Delta\mathcal{E} \tag{5.36}$$

or

$$\mathcal{E}_n + \|V_0^+\|_\infty + \lambda_+ u_1 + \Delta\mathcal{E} \le E \le \mathcal{E}_{n+1} - \|V_0^-\|_\infty - \lambda_- u_1 - 2\hat{\delta}_+\hbar\omega_c, \tag{5.37}$$

where $\mathcal{E}_{-1} = -\infty$. We call the energy interval satisfying the condition (5.36) the lower localization regime around the band center $\mathcal{E}_n$, and call the interval for the condition (5.37) the upper localization regime. These conditions imply that we cannot treat the energy E near the Landau level $\mathcal{E}_n$. From the assumption $\mathbf{A}_{\mathrm{P}} = 0$, we have

$$\mathcal{E}^{\mathrm{edge}}_{n-1,+} \le \mathcal{E}_{n-1} + \|V_0^+\|_\infty, \quad \text{and} \quad \mathcal{E}^{\mathrm{edge}}_{n,-} \ge \mathcal{E}_n - \|V_0^-\|_\infty. \tag{5.38}$$

From these bounds, one has

$$\mathcal{E}^{\mathrm{edge}}_{n-1,+} + \lambda_+ u_1 + \hat{\delta}_-\hbar\omega_c \le E \le \mathcal{E}^{\mathrm{edge}}_{n,-} - \lambda_- u_1 - \Delta\mathcal{E} \tag{5.39}$$

or

$$\mathcal{E}^{\mathrm{edge}}_{n,+} + \lambda_+ u_1 + \Delta\mathcal{E} \le E \le \mathcal{E}^{\mathrm{edge}}_{n+1,-} - \lambda_- u_1 - 2\hat{\delta}_+\hbar\omega_c, \tag{5.40}$$

and so the energy E satisfies the condition (5.5). Since the random potential V_ω satisfies the condition (4.14) on the ribbon region $\mathcal{R}$, the energy E satisfies the condition in Theorem D.7 as

$$\mathcal{E}_{n-1} + \|(V_0 + V_\omega)^+\|_\infty + \hat{\delta}_-\hbar\omega_c \le E \le \mathcal{E}_n - \|(V_0 + V_\omega)^-\|_\infty - \Delta\mathcal{E} \tag{5.41}$$

or

$$\mathcal{E}_n + \|(V_0 + V_\omega)^+\|_\infty + \Delta\mathcal{E} \le E \le \mathcal{E}_{n+1} - \|(V_0 + V_\omega)^-\|_\infty - 2\hat{\delta}_+\hbar\omega_c. \tag{5.42}$$

As a result, we can take $\beta = \tilde{\kappa}_n \ell_B^{-1} \propto \sqrt{B}$ for $B \ge B_{0,1}^{(n)}$, and the constant $C_1^{(n)}$ in (5.30) is independent of B. Here $\tilde{\kappa}_n$ and $B_{0,1}^{(n)}$ are positive constants which depend only on the index n of the Landau level.

On the other hand, one can choose δE to satisfy $(\delta E)^{-1} \sim \mathrm{Const} \times B$ for a large B from the condition (5.33) and $K_3 \sim \mathrm{Const} \times B$ for a large B. The asymptotic behavior of K_3 is easily derived from the expression (A.39). See the remark below Theorem A.2. Combining these observations with the bound (5.30), we reach the result that there exists a large, positive $B_2^{(n)}$ such that the statement (5.34) holds for all $B \geq B_2^{(n)}$. The positive constant $B_2^{(n)}$ depends only on the index n of the Landau level.

We have fixed the size ℓ of the box to $\ell = \ell_0$ at the first, and chosen a large strength B of the magnetic field. However, fixing the initial size of the box is not convenient for applying the multi-scale analysis to the present system. In fact, we must choose a sufficiently large ℓ_0 for the initial size to satisfy a certain condition which depends on the strength B of the magnetic field in the analysis. Clearly, changing the initial size ℓ_0 to a larger value is not allowed for a fixed B because the right-hand side of the bound (5.30) depends on the size ℓ_0 of the box through $(\delta E)^{-1} \propto \ell_0^{\xi+2}$. In order to avoid this technical difficulty, we take ℓ_0 to be a function of B as

$$\ell_0 = \ell_0(B) = \hat{\ell}_0[\sqrt{B/B_2^{(n)}}\,]_{\mathrm{odd}}^{\leq}, \tag{5.43}$$

where $[x]_{\mathrm{odd}}^{\leq}$ denotes the largest odd integer which is smaller than or equal to x, and $B_2^{(n)}$ is the lower bound for the strength B of the magnetic field which is determined in the above; the odd integer $\hat{\ell}_0$ is chosen so that $\ell_0(B_2^{(n)})$ satisfies the condition (5.32).

From the definition (5.43) and $\beta = \tilde{\kappa}_n \ell_B^{-1}$, one has

$$e^{-\beta r_3} \leq e^{-\tilde{\gamma}_0^{(n)} \ell_0} \tag{5.44}$$

with $\tilde{\gamma}_0^{(n)} = \tilde{\kappa}_n r_3/(\ell_{B_2^{(n)}} \hat{\ell}_0)$. Using this, (5.43) and $(\delta E)^{-1} \sim \mathrm{Const} \times B\ell_0^{\xi+2}$ for a large B, the right-hand side of the bound (5.30) is bounded from above by

$$C\ell_0^{2\xi+11} e^{-\tilde{\gamma}_0^{(n)}\ell_0} = \exp[-\{\tilde{\gamma}_0^{(n)} - (\log C)/\ell_0 - (2\xi+11)(\log \ell_0)/\ell_0\}\ell_0] \tag{5.45}$$

for a large B. Here C is the positive constant. Thus there exists a large, positive $B_0^{(n)}$ such that the statement (5.34) holds for any $B \geq B_0^{(n)}$. The positive constant $B_0^{(n)}$ depends only on the index n of the Landau level. In addition to this, we can choose the constant $\gamma_0 = \gamma_0^{(n)}$ so that $\gamma_0^{(n)}$ is independent of B and of the initial size $\ell_0 = \ell_0(B)$. Actually $\gamma_0^{(n)}$ depends only on the index n of the Landau level.

Next consider the case with $\mathbf{A}_{\mathrm{P}} \neq 0$. In this case, we also take $\ell_0 = \ell_0(B)$ of (5.43). The decay bound (D.85) for the resolvent in Theorem D.7 was the key to the above argument. However, for $\mathbf{A}_{\mathrm{P}} \neq 0$, we must rely on the different, weaker bound (D.25) in Theorem D.2. In fact, we cannot obtain a similar bound to (D.85) because of a technical reason. To begin with, let us see the difference between the two bounds. Let E be the energy in the spectrum $\sigma(H_\omega)$ of the Hamiltonian H_ω of the whole system, and let $\sigma(H_{\mathcal{R}})$ be the spectrum of the Hamiltonian $H_{\mathcal{R}}$ of (5.13) having the local potential $V_{\mathcal{R}}$ supported by the ribbon region $\mathcal{R}$. Then the distance between E and $\sigma(H_{\mathcal{R}})$ is at most of order of $\|V_\omega\|_\infty$. Namely,

$$\mathrm{dist}(\sigma(H_{\mathcal{R}}), E) = \min\{|E_+ - E|, |E - E_-|\} \leq \|V_\omega\|_\infty, \tag{5.46}$$

where (E_-, E_+) is the spectral gap of $H_{\mathcal{R}}$. Substituting this into the expression (D.24) of β in the bound (D.25) for the resolvent, we have that the parameter β is at most of $\mathcal{O}(1)$ for a large strength B of the magnetic field. Thus we cannot realize a large β by taking only a large B.

In order to realize a large β, we require a strong disorder, together with the strong magnetic field. To this end, we take $u = \hbar\omega_c \hat{u}$ with a fixed, dimensionless function $\hat{u}$ for the random potential V_ω of (2.6), and choose $C_0 = \hbar\omega_c \hat{C}_0$ with a fixed, dimensionless constant $\hat{C}_0$ as the constant in (D.6). Fix $\hat{\delta}_\pm > 0$. Assume that the energy E satisfies

$$\tilde{\mathcal{E}}_{n,+}^{\text{edge}} + \hbar\omega_c\hat{\delta}_- \le E \le \tilde{\mathcal{E}}_{n+1,-}^{\text{edge}} - \hbar\omega_c\hat{\delta}_+, \tag{5.47}$$

where

$$\tilde{\mathcal{E}}_{n,+}^{\text{edge}} = \mathcal{E}_n + \frac{\sqrt{2}e}{\sqrt{m_e}}\||\mathbf{A}_{\mathrm{P}}|\|_\infty\sqrt{\mathcal{E}_n} + \frac{e^2}{2m_e}\||\mathbf{A}_{\mathrm{P}}|\|_\infty^2 + \|V_0^+\|_\infty + \lambda_+ u_1 \tag{5.48}$$

and

$$\tilde{\mathcal{E}}_{n+1,-}^{\text{edge}} = \mathcal{E}_{n+1} - \frac{\sqrt{2}e}{\sqrt{m_e}}\||\mathbf{A}_{\mathrm{P}}|\|_\infty\sqrt{\mathcal{E}_{n+1}} - \|V_0^-\|_\infty - \lambda_- u_1. \tag{5.49}$$

The condition (5.47) implies that we cannot treat the energy E in the interval $[\tilde{\mathcal{E}}_{n,-}^{\text{edge}} - \hbar\omega_c\hat{\delta}_+, \tilde{\mathcal{E}}_{n,+}^{\text{edge}} + \hbar\omega_c\hat{\delta}_-]$ near the Landau level $\mathcal{E}_n$. From (3.7), (3.9) and (4.14), we have

$$E_- \le \mathcal{E}_{n,+}^{\text{edge}} + \lambda_+ u_1 \le \tilde{\mathcal{E}}_{n,+}^{\text{edge}} \quad \text{and} \quad E_+ \ge \mathcal{E}_{n+1,-}^{\text{edge}} - \lambda_- u_1 \ge \tilde{\mathcal{E}}_{n+1,-}^{\text{edge}}. \tag{5.50}$$

Therefore we obtain $E_+ - E \ge \hbar\omega_c\hat{\delta}_+$ and $E - E_- \ge \hbar\omega_c\hat{\delta}_-$. Substituting these into the expression (D.24) of β, we have

$$\beta \ge \frac{\sqrt{2m_e}}{\hbar}\sqrt{\frac{(\hbar\omega_c)^3\hat{C}_0\hat{\delta}_+\hat{\delta}_-}{C_0(E-E_-) + 16(E_+ + \tilde{C}_0)(E_- + \tilde{C}_0)}} = \mathcal{O}(B^{1/2}) \quad \text{for a large } B, \tag{5.51}$$

where we have used $C_0 = \hbar\omega_c\hat{C}_0$ with a fixed constant $\hat{C}_0$. In this case, one can easily have $K_3 = \mathcal{O}(1)$ for a large B, from (A.3), (A.4), (A.7), (A.25) and (A.39). Therefore we can choose $(\delta E)^{-1} \sim \text{Const} \times \ell_0^{\xi+2}$ to satisfy the condition (5.33). Consequently the same statement (5.34) holds for a strong magnetic field and for a strong potential $u = \mathcal{O}(B)$.

6 Multi-Scale Analysis

Starting from the initial decay estimates (5.34) and (5.35) for the resolvent, we derive similar estimates for larger scales without losing too much. The main results of this section are given in Lemmas 6.2 and 6.3 below. The proofs are given in Appendix F. We stress again that these results for large but finite volumes never yield a similar decay bound for two arbitrary points in the whole plane $\mathbf{R}^2$. As to the resolvent in the infinite volume, we will rely on the fractional moment method in the next section. The multi-scale analysis given here is a simplified version [31, 41, 42] of [29]. Although the method itself is well known, we must carefully handle the magnetic field dependence of the decay bound for the resolvent again.

Let ℓ be an odd integer larger than 1, and denote by $\Lambda_\ell(\mathbf{z}) = \Lambda_{\ell,\ell}^{\text{para}}(\mathbf{z})$ the parallelogram box with sidelength ℓ and with center $\mathbf{z} \in \Gamma_\ell := \ell\mathbf{L}^2 = \{m\ell\mathbf{a}_1 + n\ell\mathbf{a}_2 \mid m, n \in \mathbf{Z}\}$. The distance between two lattice sites in Γ_ℓ is defined by $|\mathbf{z}| = \max\{|m|\ell, |n|\ell\}$. Fix a small $\delta > 0$. Let $\chi_\ell(\mathbf{z})$ be the characteristic function of the region $\Lambda_\ell(\mathbf{z})$, and let $\chi_{3\ell}^\delta(\mathbf{z})$ be a C^3, positive cut-off function satisfying

$$\chi_{3\ell}^\delta(\mathbf{z})|_{\Lambda_{3\ell}^\delta(\mathbf{z})} = 1, \quad \text{and} \quad \operatorname{supp}|\nabla\chi_{3\ell}^\delta(\mathbf{z})| \subset \Lambda_{3\ell}(\mathbf{z})\backslash\Lambda_{3\ell}^\delta(\mathbf{z}), \tag{6.1}$$

where $\Lambda_{3\ell}^{\delta}(\mathbf{z}) := \{\mathbf{r} \in \Lambda_{3\ell}(\mathbf{z}) | \mathrm{dist}(\mathbf{r}, \partial\Lambda_{3\ell}(\mathbf{z}) > \delta\}$. We write

$$\begin{aligned} R_{3\ell,\mathbf{z}}(E+i\varepsilon) &= (\hat{H}_{\Lambda_{3\ell}(\mathbf{z})} - E - i\varepsilon)^{-1}, \\ \tilde{W}_{3\ell}^{\delta}(\mathbf{z}) &= W(\chi_{3\ell}^{\delta}(\mathbf{z})) - \delta V_{\omega,3\ell,3\ell}\chi_{3\ell}^{\delta}(\mathbf{z}). \end{aligned} \tag{6.2}$$

Here the Hamiltonian $\hat{H}_{\Lambda_{3\ell}(\mathbf{z})}$ is given by (5.4).

Definition 6.1 A parallelogram box $\Lambda_{3\ell}(\mathbf{z})$ is called γ-good for some $\gamma > 0$ if the following two bounds hold:

$$\sup_{\varepsilon\neq 0} \| \tilde{W}_{3\ell}^{\delta}(\mathbf{z}) R_{3\ell,\mathbf{z}}(E+i\varepsilon)\chi_{\ell}(\mathbf{z})\| \le e^{-\gamma\ell} \tag{6.3}$$

and

$$\sup_{\varepsilon\neq 0} \| \chi_{\ell}(\mathbf{z}) R_{3\ell,\mathbf{z}}(E+i\varepsilon)(\tilde{W}_{3\ell}^{\delta}(\mathbf{z}))^{*}\| \le e^{-\gamma\ell}. \tag{6.4}$$

Remark The probability Prob[$\Lambda_{3\ell}(\mathbf{z})$ is γ-good] is independent of the center $\mathbf{z}$.

Lemma 6.2 *Let ℓ, ℓ' be odd integers larger than* 1 *such that ℓ' is a multiple of ℓ and satisfies $\ell' > 4\ell$. Assume* Prob[$\Lambda_{3\ell}(\cdots)$ *is γ-good*] $\ge 1-\eta$ *with a small $\eta > 0$. Then*

$$\mathrm{Prob}[\Lambda_{3\ell'}(\cdots) \textit{ is } \gamma'\textit{-good}] \ge 1-\eta' \tag{6.5}$$

with

$$\eta' = (5\ell'/\ell)^4\eta^2 + (\ell')^{-\xi}/2 \tag{6.6}$$

and with

$$\gamma' = \gamma(1-4\ell/\ell') - \ell^{-1}\log(c_0K_3^2|E|) - (\ell')^{-1}(2s+7)\log\ell'. \tag{6.7}$$

Here c_0 is a positive constant.

We define a sequence of monotone increasing length scales ℓ_k as

$$\ell_{k+1} = \ell_k[\ell_k^{1/2}]_{\mathrm{odd}}^{\ge} \quad \text{for } k = 0, 1, 2, \ldots, \tag{6.8}$$

where $[x]_{\mathrm{odd}}^{\ge}$ denotes the smallest odd integer which is larger than or equal to x. Clearly we have $\ell_{k+1} \ge \ell_k^{3/2}$ and $\ell_{k+1} > 4\ell_k$ for all k if the initial scale ℓ_0 is large enough.

Lemma 6.3 *Take $\ell_0 = \ell_0(B)$ which is given by* (5.43), *i.e., the function of the strength B of the magnetic field. Then there exists a minimum strength $B_0 > 0$ of the magnetic field such that*

$$\mathrm{Prob}[\Lambda_{3\ell_k}(\cdots) \textit{ is } \gamma_\infty\textit{-good}] \ge 1 - (\ell_k)^{-\xi} \tag{6.9}$$

with some $\gamma_\infty > 0$ for any $B > B_0$ and for any k.

7 Fractional Moment Bound for the Resolvent

As mentioned at the beginning of the preceding section, the multi-scale analysis has the disadvantage for the decay estimate of the resolvent in the infinite volume. In order to compensate for the disadvantage, we rely on the fractional moment method. The key points of the method are that the fractional moment of the resolvent is finite due to the resonance-diffusing effect of the disorder, and satisfies a "correlation inequality" [32]. But the resolevnt itself without taking the fractional moment cannot be evaluated by the method, as we already mentioned above. The aim of this section is to obtain the decay bound (7.1) below for the fractional moment of the resolvent for the present system, following Ref. [32]. Further, the decay bound (7.1) so obtained yields the decay bound (7.17) below for the Fermi sea projection P_{F}. In the article [32], the authors showed that a decay estimate of a resolvent in the multiscale analysis yields a fractional moment bound for the resolvent. In this section, we obtain the fractional moment bound more directly from the initial decay estimate which was studied in Sect. 5. Actually, the initial decay estimate was the initial data for the multi-scale analysis in Sect. 6.

Consider the present system described by the Hamiltonian H_ω with the random potential V_ω on a finite region Λ or the infinite plane $\mathbf{R}^2$. Let $\chi_{\mathcal{A}}, \chi_{\mathcal{B}}$ be the characteristic functions of the sets $\mathcal{A}, \mathcal{B}$ with a compact support, respectively. Then the fractional moment bound for the resolvent is

$$\sup_{\varepsilon\neq 0} \mathbf{E}\|\chi_{\mathcal{A}}(E_{\mathrm{F}}+i\varepsilon-H_\omega)^{-1}\chi_{\mathcal{B}}\|^s \le \mathrm{Const}\times e^{-\mu r}, \tag{7.1}$$

where $\mathbf{E}$ is the expectation with respect to the random variables of the potential V_ω, $s\in(0,1/3)$, μ is a positive constant, and we have written $r=\mathrm{dist}(\mathcal{A},\mathcal{B})$.

We denote by $s_\ell(\mathbf{u})$ the square box centered at $\mathbf{u}=(u_1,u_2)\in\mathbf{R}^2$ with the sidelength ℓ, i.e., $s_\ell(\mathbf{u})=\{\mathbf{r}=(x,y)\in\mathbf{R}^2 \mid \max\{|x-u_1|,|y-u_2|\}\le \ell/2\}$. Consider the Hamiltonian

$$H_{s_L(\mathbf{z}_0)}=\frac{1}{2m_e}(\mathbf{p}+e\mathbf{A})^2+(V_0+V_\omega)|_{s_L(\mathbf{z}_0)} \tag{7.2}$$

on the square region $s_L(\mathbf{z}_0)$ centered at $\mathbf{z}_0\in\mathbf{L}^2$ with the sidelength L, where we impose the Dirichlet boundary conditions, and write the resolvent as

$$R_L=R_L(E+i\varepsilon)=(H_{s_L(\mathbf{z}_0)}-E-i\varepsilon)^{-1} \tag{7.3}$$

for $E,\varepsilon\in\mathbf{R}$. We write $\chi_{\tilde r}(\mathbf{z})$ for the characteristic function of the square box $s_{\tilde r}(\mathbf{z})$ with the sidelength $\tilde r:=\sqrt{3}a/2$ for $\mathbf{z}\in\mathbf{L}^2$. Let $\delta\mathcal{A}_L(\mathbf{z}_0)=s_{L-3\tilde r}(\mathbf{z}_0)\backslash s_{L-23\tilde r}(\mathbf{z}_0)$, and let $\mathbf{z}'\in\delta\mathcal{A}_L(\mathbf{z}_0)\cap\mathbf{L}^2$. In order to obtain the fractional moment bound for the resolvent, we want to evaluate

$$\sup_{\varepsilon\neq 0}\mathbf{E}[\|\chi_{\tilde r}(\mathbf{z}')R_L(E+i\varepsilon)\chi_{\tilde r}(\mathbf{z}_0)\|^s] \quad \text{for } s\in(0,1/3). \tag{7.4}$$

In the same way as in Sect. 4, one can find a ribbon region $\mathcal{R}$ such that the conditions (4.10) and (4.14) are satisfied with probability larger than

$$P^{\mathrm{perc}}=1-C^{\mathrm{perc}}Le^{-m_pL} \quad \text{with two positive constants, } C^{\mathrm{perc}} \text{ and } m_p, \tag{7.5}$$

and that the ribbon $\mathcal{R}$ encircles the square box $s_{\tilde r}(\mathbf{z}_0)$, and that the following two conditions are satisfied: $\mathrm{dist}(\mathcal{R},s_{\tilde r}(\mathbf{z}_0))>0$ and $\mathrm{dist}(\mathcal{R},s_{\tilde r}(\mathbf{z}'))>0$ for all $\mathbf{z}'\in\delta\mathcal{A}_L(\mathbf{z}_0)\cap\mathbf{L}^2$. Further, we can find a C^2, positive cut-off function χ_1 such that $\chi_1|_{s_{\tilde r}(\mathbf{z}_0)}=1$ and $\mathrm{supp}\,|\nabla\chi_1|\subset\mathcal{C}_\epsilon$, where the region $\mathcal{C}_\epsilon$ near the center $\mathcal{C}$ of the ribbon $\mathcal{R}$ is the same as in Sect. 4. Since we can choose χ_1 so that $\chi_{\tilde{r}}(\mathbf{z}')\chi_1 = 0$, we have

$$\chi_{\tilde{r}}(\mathbf{z}')R_L\chi_{\tilde{r}}(\mathbf{z}_0) = \chi_{\tilde{r}}(\mathbf{z}')R_L\chi_1\chi_{\tilde{r}}(\mathbf{z}_0) = \chi_{\tilde{r}}(\mathbf{z}')R_L W(\chi_1)R_L\chi_{\tilde{r}}(\mathbf{z}_0). \tag{7.6}$$

In the same way as in Sect. 4, we can take the C^2, positive cut-off function $\chi_{\mathcal{R}}^\epsilon$ and the resolvent $R_{\mathcal{R}}$ for the Hamiltonian $H_{\mathcal{R}}$. Therefore the right-hand side can be further written as

$$\begin{aligned}\chi_{\tilde{r}}(\mathbf{z}')R_L\chi_{\tilde{r}}(\mathbf{z}_0) &= \chi_{\tilde{r}}(\mathbf{z}')R_L W(\chi_1)R_L\chi_{\tilde{r}}(\mathbf{z}_0)\\ &= \chi_{\tilde{r}}(\mathbf{z}')R_L\chi_{\mathcal{R}}^\epsilon W(\chi_1)R_L\chi_{\tilde{r}}(\mathbf{z}_0)\\ &= -\chi_{\tilde{r}}(\mathbf{z}')R_L W(\chi_{\mathcal{R}}^\epsilon)R_{\mathcal{R}}W(\chi_1)R_L\chi_{\tilde{r}}(\mathbf{z}_0),\end{aligned} \tag{7.7}$$

where we have used $\chi_{\mathcal{R}}^\epsilon|_{\mathcal{R}\setminus\mathcal{R}_\epsilon} = 1$, $\chi_{\tilde{r}}(\mathbf{z}_0)\chi_{\mathcal{R}}^\epsilon = 0$ and the geometric resolvent equation, $R_L\chi_{\mathcal{R}}^\epsilon = \chi_{\mathcal{R}}^\epsilon R_{\mathcal{R}} - R_L W(\chi_{\mathcal{R}}^\epsilon)R_{\mathcal{R}}$. Using the bounds (5.22) and (5.23) for the resolvent R_L instead of $R_{3\ell,3\ell'}$, we obtain

$$\begin{aligned}\|\chi_{\tilde{r}}(\mathbf{z}')R_L\chi_{\tilde{r}}(\mathbf{z}_0)\| &\le \|\chi_{\tilde{r}}(\mathbf{z}')R_L W(\chi_{\mathcal{R}}^\epsilon)\chi_{\mathcal{A}}R_{\mathcal{R}}\chi_{\mathcal{B}}W(\chi_1)R_L\chi_{\tilde{r}}(\mathbf{z}_0)\|\\ &\le \|R_L W(\chi_{\mathcal{R}}^\epsilon)\|\|\chi_{\mathcal{A}}R_{\mathcal{R}}\chi_{\mathcal{B}}\|\|W(\chi_1)R_L\|\\ &\le f_1(|E|,\|R_L\|)f_2(|E|,\|R_L\|)\|\chi_{\mathcal{A}}R_{\mathcal{R}}\chi_{\mathcal{B}}\|,\end{aligned} \tag{7.8}$$

where $\mathcal{A} = \mathrm{supp}\,|\nabla\chi_{\mathcal{R}}^\epsilon|$ and $\mathcal{B} = \mathcal{C}_\epsilon \supset \mathrm{supp}\,|\nabla\chi_1|$. In the same way as in Sect. 4, we have

$$\|\chi_{\mathcal{A}}R_{\mathcal{R}}\chi_{\mathcal{B}}\| \le C_1^{(n)}e^{-\beta r_3} \tag{7.9}$$

with probability larger than $(1 - C^{\mathrm{perc}}Le^{-m_pL})$, where $C_1^{(n)}$ and β are the corresponding positive constants. By using the Wegner estimate, the norm of the resolvent can be also evaluated as $\|R_L\| \le (\delta E)^{-1}$ with probability larger than $(1 - C_W\|g\|_\infty\delta EL^2)$. We choose L to satisfy

$$C^{\mathrm{perc}}Le^{-m_pL} \le (L/a)^{-\tilde{\xi}}/2 \tag{7.10}$$

with a positive number $\tilde{\xi}$ which we will determine below, and choose δE so that

$$C_W\|g\|_\infty\delta EL^2 = (L/a)^{-\tilde{\xi}}/2. \tag{7.11}$$

Then we have

$$\|\chi_{\tilde{r}}(\mathbf{z}')R_L\chi_{\tilde{r}}(\mathbf{z}_0)\| \le C_1^{(n)}f_1(|E|,(\delta E)^{-1})f_2(|E|,(\delta E)^{-1})e^{-\beta r_3} \tag{7.12}$$

with probability larger than $(1 - (L/a)^{-\tilde{\xi}})$. We denote by D_L the set of the events ω satisfying the above inequality (7.12). Note that

$$\begin{aligned}\mathbf{E}[\|\chi_{\tilde{r}}(\mathbf{z}')R_L\chi_{\tilde{r}}(\mathbf{z}_0)\|^s] \le\ &\mathbf{E}[\|\chi_{\tilde{r}}(\mathbf{z}')R_L\chi_{\tilde{r}}(\mathbf{z}_0)\|^s\mathbf{I}(D_L)]\\ &+ \mathbf{E}[\|\chi_{\tilde{r}}(\mathbf{z}')R_L\chi_{\tilde{r}}(\mathbf{z}_0)\|^s\mathbf{I}(D_L^c)],\end{aligned} \tag{7.13}$$

where $\mathbf{I}(A)$ is the indicator function of an event A. The first term in the right-hand side is estimated as

$$\mathbf{E}[\|\chi_{\tilde{r}}(\mathbf{z}')R_L\chi_{\tilde{r}}(\mathbf{z}_0)\|^s\mathbf{I}(D_L)] \le [C_1^{(n)}f_1(|E|,(\delta E)^{-1})f_2(|E|,(\delta E)^{-1})]^s e^{-s\beta r_3}. \tag{7.14}$$

Using the Hölder inequality with $s < t < 1$, the second term is estimated as

$$\begin{aligned}\mathbf{E}[\|\chi_{\tilde r}(\mathbf{z}')R_L\chi_{\tilde r}(\mathbf{z}_0)\|^s\mathbf{I}(D_L^c)] &\le \mathbf{E}[\|\chi_{\tilde r}(\mathbf{z}')R_L\chi_{\tilde r}(\mathbf{z}_0)\|^t]^{s/t}\mathbf{E}[\mathbf{I}(D_L^c)]^{1-s/t}\\ &\le \mathrm{Const}\times B^s\times(L/a)^{-(1-s/t)\tilde\xi},\end{aligned}\tag{7.15}$$

where B is the strength of the magnetic field, and we have used the fractional moment bound (3.19) in [32], and $\mathbf{E}[\mathbf{I}(D_L^c)]\le(L/a)^{-\tilde\xi}$. The factor B^s comes from a careful but easy calculation in the fractional moment bound. The positive constant depends only on the index of the Landau level in the condition (5.5) for the energy E.

We take L to be a function of B as $L = L(B) = \mathrm{Const}\times B^{1/2}$. Then the argument of Sect. 5 yields $\beta r_3 = \mathrm{Const}\times B^{1/2}$. We choose t and $\tilde\xi$ to satisfy $(1-s/t)\tilde\xi > 3+12s$. Combining these, (7.13), (7.14) and (7.15), we obtain that the quantity,

$$B^{5s}L^3\sup_{\varepsilon\ne0}\mathbf{E}[\|\chi_{\tilde r}(\mathbf{z}')R_L(E+i\varepsilon)\chi_{\tilde r}(\mathbf{z}_0)\|^s],\tag{7.16}$$

becomes small for a sufficiently large B of the strength of the magnetic field. This implies that the finite-volume criteria[7] of Theorem 1.2 in [32] is satisfied for magnetic fields whose strength B is larger than a positive B_0. The factor B^{5s} comes from the B-dependence of the constant in (1.18) of the criteria. Thus the fractional moment bound (7.1) for the resolvent holds for such a large magnetic field.

Let P_{F} be the projection on energies smaller than the Fermi energy E_{F}, and let $\chi_{\mathcal{A}}, \chi_{\mathcal{B}}$ be the characteristic functions of the sets $\mathcal{A},\mathcal{B}$ with a compact support, respectively. The following Lemma 7.1 is due to [32]. In order to make the paper self-contained, we give a proof which is slightly different from that in [32].

Lemma 7.1 *The following bound holds*:

$$\mathbf{E}\|\chi_{\mathcal{A}}P_{\mathrm{F}}\chi_{\mathcal{B}}\|\le\mathrm{Const}\times\exp[-\mu\,\mathrm{dist}(\mathcal{A},\mathcal{B})].\tag{7.17}$$

Proof Write $R(z)=(z-H_\omega)^{-1}$. Using the contour integral, one has

$$\begin{aligned}\chi_{\mathcal{A}}P_{\mathrm{F}}\chi_{\mathcal{B}} &= \frac{1}{2\pi i}\int_{E_0}^{E_{\mathrm{F}}}dE\,\chi_{\mathcal{A}}R(E+iy_-)\chi_{\mathcal{B}}+\frac{1}{2\pi i}\int_{y_-}^{y_+}idy\,\chi_{\mathcal{A}}R(E_{\mathrm{F}}+iy)\chi_{\mathcal{B}}\\ &\quad+\frac{1}{2\pi i}\int_{E_{\mathrm{F}}}^{E_0}dE\,\chi_{\mathcal{A}}R(E+iy_+)\chi_{\mathcal{B}}+\frac{1}{2\pi i}\int_{y_+}^{y_-}idy\,\chi_{\mathcal{A}}R(E_0+iy)\chi_{\mathcal{B}},\end{aligned}\tag{7.18}$$

where E_0 is a real constant satisfying $H_\omega > E_0$. The integral near the Fermi energy is justified because the operator norm limit, $\lim_{\varepsilon\downarrow0}\chi_{\mathcal{A}}R(E\pm i\varepsilon)\chi_{\mathcal{B}}$, exists [32, 42, 43] almost surely for almost every energy $E\in\mathbf{R}$. We can choose finite E_0 and $y_\pm$ so that

$$\|\chi_{\mathcal{A}}R(E_0+iy)\chi_{\mathcal{B}}\|\le\mathrm{Const}\times e^{-\mu r}\quad\text{for any real } y\tag{7.19}$$

and

$$\|\chi_{\mathcal{A}}R(E+iy_\pm)\chi_{\mathcal{B}}\|\le\mathrm{Const}\times e^{-\mu r}\quad\text{for } E\in[E_0,E_{\mathrm{F}}]\tag{7.20}$$

[7] We should remark the following: The condition $\mathbf{z}'\in\delta\mathcal{A}_L(\mathbf{z}_0)\cap\mathbf{L}^2$ is slightly different from that in Theorem 1.2 of [32]. In fact, our argument relies on Lemma 4.1 of [32].

with the same decay constant μ. See Appendix D.1 for details. Therefore it is enough to evaluate the second integral in the right-hand side of (7.18). It is written

$$\liminf_{\varepsilon_n\to 0}\int_{I(\varepsilon_n)} dy\,\chi_{\mathcal{A}} R(E_{\mathrm{F}}+iy)\chi_{\mathcal{B}}, \tag{7.21}$$

where $\{\varepsilon_n\}_n$ is a decreasing sequence, and we have written $I(\varepsilon_n)=[y_-,y_+]\backslash(-\varepsilon_n,\varepsilon_n)$. Using Fatou's lemma, we have

$$\mathbf{E}\left\|\int_{y_-}^{y_+} dy\,\chi_{\mathcal{A}} R(E_{\mathrm{F}}+iy)\chi_{\mathcal{B}}\right\| \le \liminf_{\varepsilon_n\to 0}\mathbf{E}\left\|\int_{I(\varepsilon_n)} dy\,\chi_{\mathcal{A}} R(E_{\mathrm{F}}+iy)\chi_{\mathcal{B}}\right\|. \tag{7.22}$$

This right-hand side is evaluated as

$$\begin{aligned}
&\mathbf{E}\left\|\int_{I(\varepsilon_n)} dy\,\chi_{\mathcal{A}} R(E_{\mathrm{F}}+iy)\chi_{\mathcal{B}}\right\| \\
&\quad\le \mathbf{E}\int_{I(\varepsilon_n)} dy\,\|\chi_{\mathcal{A}} R(E_{\mathrm{F}}+iy)\chi_{\mathcal{B}}\| \\
&\quad\le \mathbf{E}\int_{I(\varepsilon_n)} dy\,\|\chi_{\mathcal{A}} R(E_{\mathrm{F}}+iy)\chi_{\mathcal{B}}\|^s\,\|\chi_{\mathcal{A}} R(E_{\mathrm{F}}+iy)\chi_{\mathcal{B}}\|^{1-s} \\
&\quad\le \mathbf{E}\int_{I(\varepsilon_n)} dy\,\|\chi_{\mathcal{A}} R(E_{\mathrm{F}}+iy)\chi_{\mathcal{B}}\|^s\,|y|^{s-1} \\
&\quad\le \mathrm{Const}\times s^{-1}(|y_+|^s+|y_-|^s)e^{-\mu r},
\end{aligned} \tag{7.23}$$

where we have used Fubini-Tonelli theorem and the decay bound (7.1) for the resolvent. This yields the desired result. □

8 Finite Volume Hall Conductance

We recall the previous results of the linear response coefficients [8]. The total conductance for finite volume and for $t\ge 0$ is written

$$\sigma_{\mathrm{tot},sy}(t)=\begin{cases}\sigma_{xy}+\gamma_{xy}\cdot t+\delta\sigma_{xy}(t), & \text{for } s=x;\\ \gamma_{yy}\cdot t+\delta\sigma_{yy}(t), & \text{for } s=y.\end{cases} \tag{8.1}$$

Our goal is to give the proof of all the statements of Theorems 2.1 and 2.2. Namely, when the Fermi level lies in the localization regime, the Hall conductance σ_{xy} is quantized to the integer as in (2.26), and both of the acceleration coefficients γ_{sy} vanish, and the corrections $\delta\sigma_{sy}(t)$ due to the initial adiabatic process are small as in the bound (2.27). For this purpose, we first treat the Hall conductance σ_{xy}, and prepare some technical lemmas for the Hall conductance σ_{xy} for finite volume in this section.

In the following, we write $\Lambda=\Lambda^{\mathrm{sys}}$ for short. The explicit form of the Hall conductance σ_{xy} for the finite region Λ is given by [8]

$$\sigma_{xy}=-\frac{i\hbar e^2}{L_xL_y}\,\mathrm{Tr}\,P_{\mathrm{F},\Lambda}[P_{x,\Lambda},P_{y,\Lambda}], \tag{8.2}$$

where $P_{\mathrm{F},\Lambda}$ is the corresponding Fermi sea projection and

$$P_{s,\Lambda} = \frac{1}{2\pi i}\int_\gamma dz R_\Lambda(z) v_s R_\Lambda(z) \quad \text{for } s = x, y. \tag{8.3}$$

Here v_s are the velocity operators, i.e., $(v_x, v_y) = \mathbf{v}(t=0)$ for $\mathbf{v}(t)$ of (2.21), and $R_\Lambda = (z - H_{\omega,\Lambda})^{-1}$ with the finite-volume Hamiltonian $H_{\omega,\Lambda}$ with the periodic boundary conditions; the closed path γ encircles the energy eigenvalues below the Fermi level E_F.

Take two rectangular regions Ω and Λ' so that the following conditions are satisfied:

$$\Omega \subset \Lambda' \subset \Lambda, \qquad \mathrm{dist}(\Omega, \partial\Lambda') = \delta L/2 \quad \text{and} \quad \mathrm{dist}(\Lambda', \partial\Lambda) = \delta L/2, \tag{8.4}$$

where we have taken the width δL of the boundary regions as $\delta L = a(L/a)^\kappa$ with $\kappa \in (1/2, 1)$ and with $L = \max\{L_x, L_y\}$. Here a is the lattice constant of the triangular lattice $\mathbf{L}^2$. Clearly we can take Ω satisfying $|\Omega| = \mathcal{O}(L^2)$ and $|\Lambda\backslash\Omega| = \mathcal{O}(L\delta L)$. We decompose σ_{xy} into two parts as $\sigma_{xy} = \sigma_{xy}^{\mathrm{in}} + \sigma_{xy}^{\mathrm{out}}$ with

$$\sigma_{xy}^{\mathrm{in}} = -\frac{i\hbar e^2}{L_x L_y}\mathrm{Tr}\,\chi_\Omega P_{\mathrm{F},\Lambda}[P_{x,\Lambda}, P_{y,\Lambda}]\chi_\Omega \tag{8.5}$$

and

$$\sigma_{xy}^{\mathrm{out}} = -\frac{i\hbar e^2}{L_x L_y}\mathrm{Tr}\,\chi_\Omega^c P_{\mathrm{F},\Lambda}[P_{x,\Lambda}, P_{y,\Lambda}]\chi_\Omega^c, \tag{8.6}$$

where χ_Ω is the characteristic function of Ω, and $\chi_\Omega^c = 1 - \chi_\Omega$. We choose k so that

$$8\ell_k a \le \delta L < 8\ell_{k+1} a, \tag{8.7}$$

where ℓ_k is a length scale in the sequence $\{\ell_k\}_k$ which is determined by the recursive equation (6.8).

Lemma 8.1 *Let $\mathcal{A}, \mathcal{B}$ be subsets of Λ. If* $\mathrm{dist}(\mathcal{A}, \mathcal{B}) \ge 7\ell_k a/2$, *then the following bound is valid*:

$$\|\chi_\mathcal{A} R_\Lambda(z)\chi_\mathcal{B}\| \le \mathrm{Const} \times L^{\kappa(\xi-2)+4}\exp[-\mu_\infty L^{2\kappa/3}] \tag{8.8}$$

with probability larger than $(1 - \mathrm{Const} \times L^{-2[\kappa(\xi+2)-3]/3})$, *where $\chi_\mathcal{A}, \chi_\mathcal{B}$ are, respectively, the characteristic functions of $\mathcal{A}, \mathcal{B}$, and μ_∞ is a positive constant.*

Proof In order to prove the statement of Lemma 8.1, we rely on the argument of the multiscale analysis in Sect. 6. Therefore we use the same notations, $\Lambda_\ell(\mathbf{z})$, $\chi_\ell(\mathbf{z})$, $\chi_{3\ell}^\delta(\mathbf{z})$, etc. From the assumption $\mathrm{dist}(\mathcal{A}, \mathcal{B}) \ge 7\ell_k a/2$, there is a sublattice $\mathbf{L}_\mathcal{B}$ of $\ell_k\mathbf{L}^2$ such that $\mathcal{B} \subset \bigcup_{\mathbf{u}\in\mathbf{L}_\mathcal{B}} \Lambda_{\ell_k}(\mathbf{u})$ and that $\mathrm{dist}(\mathcal{A}, \Lambda_{3\ell_k}(\mathbf{u})) > 0$ for all $\mathbf{u} \in \mathbf{L}_\mathcal{B}$. Using the adjoint of the geometric resolvent equation,

$$R_\Lambda(z)\chi_{3\ell_k}^\delta(\mathbf{u}) = \chi_{3\ell_k}^\delta(\mathbf{u})R_{3\ell_k,\mathbf{u}}(z) + R_\Lambda(z)(\tilde{W}_{3\ell_k}^\delta(\mathbf{u}))^* R_{3\ell_k,\mathbf{u}}(z), \tag{8.9}$$

we have

$$\begin{aligned}\chi_\mathcal{A} R_\Lambda(z)\chi_{\ell_k}(\mathbf{u}) &= \chi_\mathcal{A}\chi_{3\ell_k}^\delta(\mathbf{u})R_{3\ell_k,\mathbf{u}}(z)\chi_{\ell_k}(\mathbf{u}) + \chi_\mathcal{A} R_\Lambda(z)(\tilde{W}_{3\ell_k}^\delta(0))^* R_{3\ell_k,\mathbf{u}}(z)\chi_{\ell_k}(\mathbf{u})\\ &= \chi_\mathcal{A} R_\Lambda(z)(\tilde{W}_{3\ell_k}^\delta(\mathbf{u}))^* R_{3\ell_k,\mathbf{u}}(z)\chi_{\ell_k}(\mathbf{u})\end{aligned} \tag{8.10}$$

for $\mathbf{u} \in \mathbf{L}_{\mathcal{B}}$, where we have used $\chi_{\mathcal{A}} \chi_{3\ell_k}^{\delta}(\mathbf{u}) = 0$ which follows from $\mathrm{dist}(\mathcal{A}, \Lambda_{3\ell_k}(\mathbf{u})) > 0$. This yields

$$\|\chi_{\mathcal{A}} R_\Lambda(z) \chi_{\ell_k}(\mathbf{u})\| \le \|R_\Lambda(z)\| \|(\tilde{W}_{3\ell_k}^{\delta}(\mathbf{u}))^* R_{3\ell_k,\mathbf{u}} \chi_{\ell_k}(\mathbf{u})\|. \tag{8.11}$$

On the other hand, we can prove the bound which is given by replacing the operator $\tilde{W}_{3\ell'}^{\delta}(\mathbf{u})$ with its adjoint in the bound (F.24) in the same way as in the proof of Lemma F.3. Therefore the argument of Lemma 6.3 yields

$$\|(\tilde{W}_{3\ell_k}^{\delta}(\mathbf{u}))^* R_{3\ell_k,\mathbf{u}} \chi_{\ell_k}(\mathbf{u})\| \le e^{-\gamma_\infty \ell_k} \tag{8.12}$$

with the probability larger than $(1 - \ell_k^{-\xi})$. From the Wegner estimate (A.38), one has $\|R_\Lambda\| \le C_W K_3 \|g\|_\infty \ell_k^{\xi} |\Lambda|$ with probability larger than $(1 - \ell_k^{-\xi})$. From (F.37), we have $\delta L < 8\ell_{k+1} a < 16\ell_k^{3/2} a$. Immediately, $(\delta L/16a)^{2/3} \le \ell_k$. Substituting these inequalities into (8.11), we have

$$\|\chi_{\mathcal{A}} R_\Lambda(z) \chi_{\ell_k}(\mathbf{u})\| \le \mathrm{Const} \times \ell_k^{\xi} L^2 \exp[-\mu_\infty L^{2\kappa/3}] \tag{8.13}$$

with probability larger than $(1 - 2\ell_k^{-\xi})$, where μ_∞ is the corresponding positive constant. The set $\mathcal{B}$ is covered by the sets $\Lambda_{\ell_k}(\mathbf{u})$. The number $|\mathbf{L}_{\mathcal{B}}|$ is at most $\mathcal{O}(L^2/\ell_k^2)$. Let M_Λ denote the event that the bound (8.13) holds for all of the site $\mathbf{u} \in \ell_k \mathbf{L}$ satisfying $\Lambda_{\ell_k} \cap \Lambda \ne \emptyset$. Clearly the probability $\mathrm{Prob}(M_\Lambda)$ that the event M_Λ occurs, is larger than $(1 - \mathrm{Const} \times \ell_k^{-\xi} L^2 \ell_k^{-2})$. From these observations, one can easily show the statement of the lemma. □

Let δ be a small positive number, and define $\Lambda^\delta := \{\mathbf{r} \in \Lambda \mid \mathrm{dist}(\mathbf{r}, \partial\Lambda) > \delta\}$. Then one has $\Lambda \backslash \Lambda^{\delta/2} = \{\mathbf{r} \in \Lambda \mid \mathrm{dist}(\mathbf{r}, \partial\Lambda) \le \delta/2\}$. Let $\chi_\Lambda^\delta \in C^2(\Lambda)$ be a positive cutoff function satisfying the following two conditions:

$$\chi_\Lambda^\delta|_{\Lambda^\delta} = 1 \quad \text{and} \quad \chi_\Lambda^\delta|_{\Lambda\backslash\Lambda^{\delta/2}} = 0. \tag{8.14}$$

Lemma 8.2 *The Hall conductance for the bulk region is written*

$$\sigma_{xy}^{\mathrm{in}} = \frac{e^2}{h} \frac{2\pi i}{L_x L_y} \mathrm{Tr}\, \chi_\Omega P_{\mathrm{F},\Lambda} [[P_{\mathrm{F},\Lambda}, x], [P_{\mathrm{F},\Lambda}, y]] \chi_\Omega + \mathcal{O}(\exp[-\mu'_\infty L^{2\kappa/3}]) \tag{8.15}$$

with probability larger than $(1 - \mathrm{Const} \times L^{-2[\kappa(\xi+2)-3]/3})$, *where* μ'_∞ *is a positive constant.*

Remark Since $\kappa(\xi+2) - 3 > 0$ from their definitions, the Hall conductance $\sigma_{xy}^{\mathrm{in}}$ for the bulk region in the infinite volume limit is given by

$$\sigma_{xy}^{\mathrm{in}} = \frac{e^2}{h} \lim_{L\uparrow\infty} \mathcal{I}(P_{\mathrm{F},\Lambda}; \Omega), \tag{8.16}$$

with probability one if the limit in the right-hand side exists. Here we have written

$$\mathcal{I}(P_{\mathrm{F},\Lambda}; \Omega) = \frac{2\pi i}{|\Omega|} \mathrm{Tr}\, \chi_\Omega P_{\mathrm{F},\Lambda} [[P_{\mathrm{F},\Lambda}, x], [P_{\mathrm{F},\Lambda}, y]] \chi_\Omega. \tag{8.17}$$

Proof Using the contour integral representation as in (8.3), one has

$$\mathrm{Tr}\, \chi_\Omega P_{\mathrm{F},\Lambda} P_{x,\Lambda} P_{y,\Lambda} = \frac{1}{(2\pi i)^2} \int_\gamma dz_1 \int_\gamma dz_2\, \mathrm{Tr}\, \chi_\Omega R_\Lambda(z_1) R_\Lambda(z_2) v_x R_\Lambda(z_2) P_{y,\Lambda} \chi_\Omega. \tag{8.18}$$

The integrand is decomposed into two parts as

$$\begin{aligned}
&\operatorname{Tr} \chi_\Omega R_\Lambda(z_1) R_\Lambda(z_2) v_x R_\Lambda(z_2) P_{y,\Lambda} \chi_\Omega \\
&\quad = \operatorname{Tr} \chi_\Omega R_\Lambda(z_1) R_\Lambda(z_2) v_x \chi_\Lambda^\delta R_\Lambda(z_2) P_{y,\Lambda} \chi_\Omega \\
&\qquad + \operatorname{Tr} \chi_\Omega R_\Lambda(z_1) R_\Lambda(z_2) v_x (1-\chi_\Lambda^\delta) R_\Lambda(z_2) P_{y,\Lambda} \chi_\Omega . \qquad (8.19)
\end{aligned}$$

For the second term in the right-hand side, we have

$$\begin{aligned}
&\|\chi_\Omega R_\Lambda(z_1) R_\Lambda(z_2) v_x (1-\chi_\Lambda^\delta) R_\Lambda(z_2)\| \\
&\quad \le \|\chi_\Omega R_\Lambda(z_1)\| \|\chi_{\Lambda'} R_\Lambda(z_2) v_x (1-\chi_\Lambda^\delta) R_\Lambda(z_2)\| \\
&\qquad + \|\chi_\Omega R_\Lambda(z_1)(1-\chi_{\Lambda'})\| \|R_\Lambda(z_2) v_x (1-\chi_\Lambda^\delta) R_\Lambda(z_2)\|. \qquad (8.20)
\end{aligned}$$

From (5.18), (8.4), (8.7) and Lemma 8.1, this gives the small contribution. From this and the contour integral representation, it is sufficient to consider

$$\operatorname{Tr} \chi_\Omega R_\Lambda(z_1) R_\Lambda(z_2) v_x \chi_\Lambda^\delta R_\Lambda(z_2) R_\Lambda(z_3) v_y \chi_\Lambda^\delta R_\Lambda(z_3) \chi_\Omega . \qquad (8.21)$$

Using the identity, $v_x \chi_\Lambda^\delta = (i/\hbar)[H_\omega, x]\chi_\Lambda^\delta$, one has

$$R_\Lambda(z_2) v_x \chi_\Lambda^\delta R_\Lambda(z_2) = \frac{i}{\hbar}[R_\Lambda(z_2), x\chi_\Lambda^\delta] - \frac{i}{\hbar} R_\Lambda(z_2) x W(\chi_\Lambda^\delta) R_\Lambda(z_2). \qquad (8.22)$$

In the same way, the second term in the right-hand side leads to the small correction. The statement of the lemma follows from these observations. □

We denote by $\mathbf{Z}_b^2$ the rectangular lattice $\{(b_1 n_1, b_2 n_2) \mid (n_1, n_2) \in \mathbf{Z}^2\}$ with a pair $b = (b_1, b_2)$ of lattice constants, and denote by $(\mathbf{Z}_b^2)^*$ the dual lattice, i.e., $(\mathbf{Z}_b^2)^* = \mathbf{Z}_b^2 - (b_1, b_2)/2$. Let $s_b(\mathbf{u})$ be the $b_1 \times b_2$ rectangular box centered at $\mathbf{u} = (u_1, u_2) \in \mathbf{Z}_b^2$, and $\chi_b(\mathbf{u})$ the characteristic function of $s_b(\mathbf{u})$. When we consider the characteristic function $\chi_b(\mathbf{u})$ on the region Λ, we restrict $\chi_b(\mathbf{u})$ to Λ.

Lemma 8.3 *Let $\mathbf{u} \in \mathbf{Z}_b^2$ satisfying $s_b(\mathbf{u}) \cap \Lambda \neq \emptyset$. Then there exists a positive constant C which is independent of the location $\mathbf{u}$ and of the size $|\Lambda|$ such that*

$$\mathbf{E}[|\operatorname{Tr} \chi_b(\mathbf{u}) P_{\mathrm{F},\Lambda} [P_{\mathrm{F},\Lambda}, \sharp][P_{\mathrm{F},\Lambda}, \sharp] \chi_b(\mathbf{u})|] < C, \qquad (8.23)$$

where $\sharp$ is either x or y.

Proof Note that

$$\begin{aligned}
&\mathbf{E}[|\operatorname{Tr} \chi_b(\mathbf{u}) P_{\mathrm{F},\Lambda} [P_{\mathrm{F},\Lambda}, x][P_{\mathrm{F},\Lambda}, y] \chi_b(\mathbf{u})|] \\
&\quad \le \sum_{\substack{\mathbf{v},\mathbf{w} \in \mathbf{Z}_b^2: \\ s_b(\mathbf{v}) \cap \Lambda \neq \emptyset, s_b(\mathbf{w}) \cap \Lambda \neq \emptyset}} \mathbf{E}[|\operatorname{Tr} \chi_b(\mathbf{u}) P_{\mathrm{F},\Lambda} \chi_b(\mathbf{v}) [P_{\mathrm{F},\Lambda}, x] \chi_b(\mathbf{w}) [P_{\mathrm{F},\Lambda}, y] \chi_b(\mathbf{u})|]. \qquad (8.24)
\end{aligned}$$

In order to estimate the summand in this right-hand side, we introduce the two component function (x^b, y^b) of $\mathbf{r}$ which is defined by $(x^b, y^b) = (u_1, u_2)$ for $\mathbf{r} \in s_b(\mathbf{u})$ with $\mathbf{u} \in \mathbf{Z}_b^2$.

Using this function, one has

$$\begin{aligned}
[P_{\mathrm{F},\Lambda}, x] &= [P_{\mathrm{F},\Lambda}, (x - x^b)] + [P_{\mathrm{F},\Lambda}, x^b], \\
[P_{\mathrm{F},\Lambda}, y] &= [P_{\mathrm{F},\Lambda}, (y - y^b)] + [P_{\mathrm{F},\Lambda}, y^b].
\end{aligned} \tag{8.25}$$

Further we note that, for any bounded operator A,

$$\begin{aligned}
|\operatorname{Tr} \chi_b(\mathbf{u}) P_{\mathrm{F},\Lambda} \chi_b(\mathbf{v}) A| &\le \sqrt{\operatorname{Tr} \chi_b(\mathbf{u}) P_{\mathrm{F},\Lambda} \chi_b(\mathbf{u})} \cdot \sqrt{\operatorname{Tr} A A^* \chi_b(\mathbf{v}) P_{\mathrm{F},\Lambda} \chi_b(\mathbf{v})} \\
&\le \|A\| \sqrt{\operatorname{Tr} \chi_b(\mathbf{u}) P_{\mathrm{F},\Lambda} \chi_b(\mathbf{u})} \cdot \sqrt{\operatorname{Tr} \chi_b(\mathbf{v}) P_{\mathrm{F},\Lambda} \chi_b(\mathbf{v})} \\
&\le \mathrm{Const} \times \|A\|,
\end{aligned} \tag{8.26}$$

where we have used

$$\begin{aligned}
\operatorname{Tr} \chi_\varepsilon(\mathbf{u}) P_{\mathrm{F}} \chi_\varepsilon(\mathbf{u}) &= \operatorname{Tr} \chi_\varepsilon(\mathbf{u}) \frac{1}{H_0 + \mathcal{C}_0} (H_0 + \mathcal{C}_0) P_{\mathrm{F}} (H_0 + \mathcal{C}_0) \frac{1}{H_0 + \mathcal{C}_0} \chi_\varepsilon(\mathbf{u}) \\
&= \operatorname{Tr} \chi_\varepsilon(\mathbf{u}) \frac{1}{H_0 + \mathcal{C}_0} (H_\omega - V_\omega + \mathcal{C}_0) P_{\mathrm{F}} (H_\omega - V_\omega + \mathcal{C}_0) \frac{1}{H_0 + \mathcal{C}_0} \chi_\varepsilon(\mathbf{u}) \\
&\le \mathrm{Const} \times \operatorname{Tr} \chi_\varepsilon(\mathbf{u}) \left(\frac{1}{H_0 + \mathcal{C}_0} \right)^2 \chi_\varepsilon(\mathbf{u}) < \infty.
\end{aligned} \tag{8.27}$$

Here we have taken a real number $\mathcal{C}_0$ satisfying $H_0 + \mathcal{C}_0 > 0$, and used $\|H_\omega P_{\mathrm{F}}\| < \infty$. Therefore the statement of the lemma follows from the norm bounds such as

$$\|\chi_b(\mathbf{v})[P_{\mathrm{F},\Lambda}, x^b]\chi_b(\mathbf{w})\| \le |w_1 - v_1| \|\chi_b(\mathbf{v}) P_{\mathrm{F},\Lambda} \chi_b(\mathbf{w})\|, \tag{8.28}$$

$$\|\chi_b(\mathbf{v})[P_{\mathrm{F},\Lambda}, (x - x^b)]\chi_b(\mathbf{w})\| \le \mathrm{Const} \times \|\chi_b(\mathbf{v}) P_{\mathrm{F},\Lambda} \chi_b(\mathbf{w})\| \tag{8.29}$$

and

$$\begin{aligned}
&\mathbf{E}[\|\chi_b(\mathbf{v}) P_{\mathrm{F},\Lambda} \chi_b(\mathbf{w})\| \|\chi_b(\mathbf{w}) P_{\mathrm{F},\Lambda} \chi_b(\mathbf{u})\|] \\
&\quad \le \{\mathbf{E}[\|\chi_b(\mathbf{v}) P_{\mathrm{F},\Lambda} \chi_b(\mathbf{w})\|^2]\}^{1/2} \{\mathbf{E}[\|\chi_b(\mathbf{w}) P_{\mathrm{F},\Lambda} \chi_b(\mathbf{u})\|^2]\}^{1/2} \\
&\quad \le \mathrm{Const} \times e^{-\mu|\mathbf{v}-\mathbf{w}|/2} e^{-\mu|\mathbf{w}-\mathbf{u}|/2},
\end{aligned} \tag{8.30}$$

where we have used Schwarz's inequality, $\|\chi_b(\mathbf{v}) P_{\mathrm{F},\Lambda} \chi_b(\mathbf{w})\| \le 1$, and the bound (7.17) for the Fermi sea projection. $\square$

In the same way, we obtain

Lemma 8.4 *Let* $\mathbf{u}, \mathbf{v} \in \mathbf{Z}_b^2$ *satisfying* $s_b(\mathbf{u}) \cap \Lambda \ne \emptyset$ *and* $s_b(\mathbf{v}) \cap \Lambda \ne \emptyset$. *Then there exists a positive constant* C *which is independent of the locations* $\mathbf{u}, \mathbf{v}$ *and of the size* $|\Lambda|$ *such that*

$$\mathbf{E}[|\operatorname{Tr} \chi_b(\mathbf{u}) P_{\mathrm{F},\Lambda}[P_{\mathrm{F},\Lambda}, \sharp][P_{\mathrm{F},\Lambda}, \sharp]\chi_b(\mathbf{u}) \cdot \operatorname{Tr} \chi_b(\mathbf{v}) P_{\mathrm{F},\Lambda}[P_{\mathrm{F},\Lambda}, \sharp][P_{\mathrm{F},\Lambda}, \sharp]\chi_b(\mathbf{v})|] < C, \tag{8.31}$$

where $\sharp$ *is either* x *or* y.

Using the magnetic translations and the argument in the proof of Lemma 8.2, the Hall conductance for the boundary region is written as

$$\sigma_{xy}^{\mathrm{out}} = \frac{e^2}{h} \frac{2\pi i}{L_x L_y} \sum_{\Omega'} \operatorname{Tr} \chi_{\Omega'} P'_{\mathrm{F},\Lambda}[[P'_{\mathrm{F},\Lambda}, x], [P'_{\mathrm{F},\Lambda}, y]] \chi_{\Omega'} + \mathcal{O}(\exp[-\mu'_\infty L^{2\kappa/3}]) \tag{8.32}$$

with probability larger than $(1 - \text{Const} \times L^{-2[\kappa(\xi+2)-3]/3})$, where Ω' is the translate of a portion of the boundary region $\Lambda\backslash\Omega$, and $P'_{\mathrm{F},\Lambda}$ is the corresponding translate of the Fermi sea projection. From this result, the Hall conductance $\sigma_{xy}^{\mathrm{out}}$ for the boundary region is vanishing in a limit $L \uparrow \infty$ with probability one:

Theorem 8.5 *There exists a sequence $\{L_n\}_n$ of the system sizes $L = L_n$ such that $|\sigma_{xy}^{\mathrm{out}}| \to 0$ as $L_n \to \infty$ for almost every ω.*

Proof Write

$$I'(L) = \frac{1}{L_x L_y} \sum_{\Omega'} |\operatorname{Tr} \chi_{\Omega'} P'_{\mathrm{F},\Lambda}[[P'_{\mathrm{F},\Lambda}, x], [P'_{\mathrm{F},\Lambda}, y]]\chi_{\Omega'}|. \tag{8.33}$$

By Lemma 8.3, we have $\mathbf{E}[I'(L)] \to 0$ as $L \to \infty$. Combining this and the inequality $\mathbf{E}[I'(L)] \ge \varepsilon \text{Prob}(I'(L) > \varepsilon)$, we can find a sequence $\{L_n\}_n$ of the system sizes $L = L_n$ such that $\{L_n\}_n$ satisfies the following two conditions:

$$\sum_n \text{Prob}(I'(L_n) > \varepsilon_n) < \infty \quad \text{and} \quad \sum_n L_n^{-2[\kappa(\xi+2)-3]/3} < \infty, \tag{8.34}$$

where $\{\varepsilon_n\}_n$ is a sequence satisfying $\varepsilon_n \to 0$ as $n \to \infty$. The application of Borel-Cantelli theorem yields that for almost every ω, there exists a number $n_0(\omega)$ which may depend on ω such that $I'(L_n) \le \varepsilon_n$ for all $n \ge n_0(\omega)$ and that the finite size correction for $\sigma_{xy}^{\mathrm{out}}$ is evaluated by $\mathcal{O}(\exp[-\mu'_\infty L_n^{2\kappa/3}])$ for all $n \ge n_0(\omega)$. By combining this with the expression (8.32) of $\sigma_{xy}^{\mathrm{out}}$, the statement of the theorem is proved. □

9 Integrality of the Hall Conductance—Index Theoretical Approach

In this section, integrality of the Hall conductance is proved by using the index theoretical method [10, 11, 20, 21].

When we apply the method of [10, 11] using a pair index of two projections to a concrete example of a continuous random model such as the present system, there arises a problem that we need a decay bound for the integral kernel of the Fermi sea projection whose Fermi energy lies in the localization regime. But getting such a decay bound is very difficult, and so this problem is still unsolved. Recently, Germinet, Klein and Schenker [25] proved the constancy of the Hall conductance for a random Landau Hamiltonian which is translation ergodic, without relying on a decay bound for the integral kernel of the Fermi sea projection. In their proof, they used a consequence of the multiscale analysis which is related to multiplicity of the eigenvalues of the Hamiltonian, for the Hall conductance formula[8] which is expressed in terms of switch functions instead of the position operator of the electron. This Hall conductance formula was justified [7] within the linear response approximation under the assumption on a spectral gap above the Fermi level. The integer of the quantized value of the Hall conductance can be determined under the assumption that the disordered-broadened Landau bands are disjoint, i.e., there exists a nonvanishing spectral gap between two neighboring Landau bands.

[8]The explicit form of the Hall conductance formula using switch functions is given in Appendix I. We also discuss the relation between this and the standard Hall conductance formula using the position operator instead of the switch functions.

In our approach, we assume neither the above disjoint condition for the Landau bands nor the periodicity of the potentials V_0 and $\mathbf{A}_{\mathrm{P}}$ which implies a translation ergodic Hamiltonian. But we must require the "covering condition" (2.10) which is not required in [25]. This condition is needed to estimate the number of the localized states and to obtain a decay bound [32] for a fractional moment of the resolvent. In order to circumvent the above problem about a decay estimate for the integral kernel of the Fermi sea projection, we introduce a partition of unity which is a collection of the characteristic functions of a small rectangular boxes.

Let $s_\varepsilon(\mathbf{u})$ be the $\varepsilon_1 \times \varepsilon_2$ rectangular box centered at $\mathbf{u} \in (\mathbf{Z}_\varepsilon^2)^*$ with the pair $\varepsilon = (\varepsilon_1, \varepsilon_2)$ of the sidelengths, and $\chi_\varepsilon(\mathbf{u})$ the characteristic function. We take $N_1\varepsilon_1 = a$ and $N_2\varepsilon_2 = \sqrt{3}a/2$ with large positive integers N_1, N_2 and with the lattice constant a of the triangular lattice $\mathbf{L}^2$ so that the set of all the boxes $s_\varepsilon(\mathbf{u})$ is invariant under the lattice translations of the triangular lattice $\mathbf{L}^2$. When we take the limit $\varepsilon_1, \varepsilon_2 \downarrow 0$, we keep the ratio $\varepsilon_1/\varepsilon_2$ finite. We introduce a unitary operator,

$$U_{\mathbf{a}}^\varepsilon = \sum_{\mathbf{u}\in(\mathbf{Z}_\varepsilon^2)^*} \chi_\varepsilon(\mathbf{u}) \exp[i\theta_{\mathbf{a}}(\mathbf{u})] \quad \text{for } \mathbf{a} \in \mathbf{Z}_\varepsilon^2, \tag{9.1}$$

where $\theta_{\mathbf{a}}(\mathbf{u})$ is the angle of sight from $\mathbf{a}$ to $\mathbf{u}$, i.e., $\arg(\mathbf{u}-\mathbf{a})$ in the terminology of the complex plane. Consider the operator, $T := P_{\mathrm{F}} - U_{\mathbf{a}}^\varepsilon P_{\mathrm{F}} (U_{\mathbf{a}}^\varepsilon)^*$.

Lemma 9.1 *For fixed parameters* $\varepsilon_1, \varepsilon_2$, $\mathbf{E}[\mathrm{Tr}\,|T|^3] < \infty$.

Proof Note that

$$\begin{aligned} T = P_{\mathrm{F}} - U_{\mathbf{a}}^\varepsilon P_{\mathrm{F}} (U_{\mathbf{a}}^\varepsilon)^* &= \sum_{\mathbf{u},\mathbf{v}\in(\mathbf{Z}_\varepsilon^2)^*} \chi_\varepsilon(\mathbf{u})[P_{\mathrm{F}} - U_{\mathbf{a}}^\varepsilon P_{\mathrm{F}} (U_{\mathbf{a}}^\varepsilon)^*]\chi_\varepsilon(\mathbf{v}) \\ &= \sum_{\mathbf{u},\mathbf{v}\in(\mathbf{Z}_\varepsilon^2)^*} [1 - e^{i\theta_{\mathbf{a}}(\mathbf{u}) - i\theta_{\mathbf{a}}(\mathbf{v})}]\chi_\varepsilon(\mathbf{u}) P_{\mathrm{F}} \chi_\varepsilon(\mathbf{v}). \end{aligned} \tag{9.2}$$

Define $t_{\mathbf{u},\mathbf{v}} := 1 - e^{i\theta_{\mathbf{a}}(\mathbf{u}) - i\theta_{\mathbf{a}}(\mathbf{v})}$ and $T_{\mathbf{u},\mathbf{v}} := t_{\mathbf{u},\mathbf{v}} \chi_\varepsilon(\mathbf{u}) P_{\mathrm{F}} \chi_\varepsilon(\mathbf{v})$. Following the idea of [21], we introduce $T_{\mathbf{u},\mathbf{v}}^{(\mathbf{b})} = T_{\mathbf{u},\mathbf{v}} \delta_{\mathbf{u}-\mathbf{b},\mathbf{v}}$. Clearly, one has $\sum_{\mathbf{b}} T_{\mathbf{u},\mathbf{v}}^{(\mathbf{b})} = T_{\mathbf{u},\mathbf{v}}$ and

$$\begin{aligned} (T^{(\mathbf{b})*} T^{(\mathbf{b})})_{\mathbf{u},\mathbf{v}} &= \sum_{\mathbf{w}} T_{\mathbf{w},\mathbf{u}}^* \delta_{\mathbf{w}-\mathbf{b},\mathbf{u}} T_{\mathbf{w},\mathbf{v}} \delta_{\mathbf{w}-\mathbf{b},\mathbf{v}} \\ &= T_{\mathbf{u}+\mathbf{b},\mathbf{u}}^* T_{\mathbf{u}+\mathbf{b},\mathbf{u}} \delta_{\mathbf{u},\mathbf{v}} \\ &= |t_{\mathbf{u}+\mathbf{b},\mathbf{u}}|^2 \chi_\varepsilon(\mathbf{u}) P_{\mathrm{F}} \chi_\varepsilon(\mathbf{u}+\mathbf{b}) P_{\mathrm{F}} \chi_\varepsilon(\mathbf{u}) \delta_{\mathbf{u},\mathbf{v}}. \end{aligned} \tag{9.3}$$

Using these identities and Minkowski's inequality, one obtains

$$\begin{aligned} (\mathbf{E}\,\mathrm{Tr}\,|T|^3)^{1/3} &\le \sum_{\mathbf{b}} (\mathbf{E}\,\mathrm{Tr}\,|T^{(\mathbf{b})}|^3)^{1/3} \\ &\le \sum_{\mathbf{b}} \left\{ \sum_{\mathbf{u}} |t_{\mathbf{u}+\mathbf{b},\mathbf{u}}|^3 \mathbf{E}[\mathrm{Tr}\,|\chi_\varepsilon(\mathbf{u}) P_{\mathrm{F}} \chi_\varepsilon(\mathbf{u}+\mathbf{b}) P_{\mathrm{F}} \chi_\varepsilon(\mathbf{u})|^{3/2}] \right\}^{1/3}. \end{aligned} \tag{9.4}$$

Since the inequality,

$$|1 - e^{i\theta_{\mathbf{a}}(\mathbf{u}) - i\theta_{\mathbf{a}}(\mathbf{v})}| \le \frac{2|\mathbf{u}-\mathbf{v}|}{|\mathbf{u}-\mathbf{a}|}, \tag{9.5}$$

holds as in [10, 11], one obtains

$$\sum_{\mathbf{u}} |t_{\mathbf{u}+\mathbf{b},\mathbf{u}}|^3 \le 2^3 \sum_{\mathbf{u}} \frac{|\mathbf{b}|^3}{|\mathbf{u}-\mathbf{a}|^3} \le \text{Const} \times |\mathbf{b}|^3. \tag{9.6}$$

Note that

$$\begin{aligned}
&\operatorname{Tr} |\chi_\varepsilon(\mathbf{u}) P_{\mathrm{F}} \chi_\varepsilon(\mathbf{u}+\mathbf{b}) P_{\mathrm{F}} \chi_\varepsilon(\mathbf{u})|^{3/2} \\
&\quad \le \sqrt{\operatorname{Tr} \chi_\varepsilon(\mathbf{u}) P_{\mathrm{F}} \chi_\varepsilon(\mathbf{u}+\mathbf{b}) P_{\mathrm{F}} \chi_\varepsilon(\mathbf{u})} \\
&\qquad \times \sqrt{\operatorname{Tr} \chi_\varepsilon(\mathbf{u}) P_{\mathrm{F}} \chi_\varepsilon(\mathbf{u}+\mathbf{b}) P_{\mathrm{F}} \chi_\varepsilon(\mathbf{u}) P_{\mathrm{F}} \chi_\varepsilon(\mathbf{u}+\mathbf{b}) P_{\mathrm{F}} \chi_\varepsilon(\mathbf{u})} \\
&\quad \le \sqrt{\operatorname{Tr} \chi_\varepsilon(\mathbf{u}) P_{\mathrm{F}} \chi_\varepsilon(\mathbf{u})} \cdot \| \chi_\varepsilon(\mathbf{u}) P_{\mathrm{F}} \chi_\varepsilon(\mathbf{u}+\mathbf{b}) \| \sqrt{\operatorname{Tr} \chi_\varepsilon(\mathbf{u}) P_{\mathrm{F}} \chi_\varepsilon(\mathbf{u})} \\
&\quad \le \| \chi_\varepsilon(\mathbf{u}) P_{\mathrm{F}} \chi_\varepsilon(\mathbf{u}+\mathbf{b}) \| \operatorname{Tr} \chi_\varepsilon(\mathbf{u}) P_{\mathrm{F}} \chi_\varepsilon(\mathbf{u}).
\end{aligned} \tag{9.7}$$

From this, the decay bound (7.17) for the Fermi sea projection and (8.27), we have

$$\mathbf{E}[\operatorname{Tr} |\chi_\varepsilon(\mathbf{u}) P_{\mathrm{F}} \chi_\varepsilon(\mathbf{u}+\mathbf{b}) P_{\mathrm{F}} \chi_\varepsilon(\mathbf{u})|^{3/2}] \le \text{Const} \times e^{-\mu|\mathbf{b}|}. \tag{9.8}$$

Combining this, (9.4) and (9.6) yields

$$(\mathbf{E} \operatorname{Tr} |T|^3)^{1/3} \le \text{Const} \times \sum_{\mathbf{b}} |\mathbf{b}| e^{-\mu|\mathbf{b}|/3} < \infty. \tag{9.9}$$

□

The result implies that the operator T^3 is trace class for almost every ω. Thus we can define the relative index [10, 11],

$$\operatorname{Index}(P_{\mathrm{F}}, U_{\mathbf{a}}^\varepsilon P_{\mathrm{F}} (U_{\mathbf{a}}^\varepsilon)^*) := \operatorname{Tr}(P_{\mathrm{F}} - U_{\mathbf{a}}^\varepsilon P_{\mathrm{F}} (U_{\mathbf{a}}^\varepsilon)^*)^3, \tag{9.10}$$

for the pair of the projections. This right-hand side takes an integer value as proved in [10, 11].

Let ℓ_{P} be a large positive integer. Let $\mathbf{A}^{\mathrm{LP}} \in C^1(\mathbf{R}^2)$ be a periodic vector potential satisfying the periodicity,

$$\mathbf{A}^{\mathrm{LP}}(\mathbf{r} + \ell_{\mathrm{P}} \mathbf{a}_1) = \mathbf{A}^{\mathrm{LP}}(\mathbf{r} + \ell_{\mathrm{P}} \mathbf{a}_2) = \mathbf{A}^{\mathrm{LP}}(\mathbf{r}), \tag{9.11}$$

and V_0^{LP} a periodic electrostatic potential satisfying the same periodicity,

$$V_0^{\mathrm{LP}}(\mathbf{r} + \ell_{\mathrm{P}} \mathbf{a}_1) = V_0^{\mathrm{LP}}(\mathbf{r} + \ell_{\mathrm{P}} \mathbf{a}_2) = V_0^{\mathrm{LP}}(\mathbf{r}), \tag{9.12}$$

where $\mathbf{a}_j$ are the primitive translation vectors of the triangular lattice $\mathbf{L}^2$. In order to prove integrality of the Hall conductance, we consider the Hamiltonian,

$$H_\omega^{\mathrm{LP}} = \frac{1}{2m_e} [\mathbf{p} + e(\mathbf{A}^{\mathrm{LP}} + \mathbf{A}_0)]^2 + V_0^{\mathrm{LP}} + V_\omega, \tag{9.13}$$

on the whole plane $\mathbf{R}^2$. Namely this Hamiltonian is obtained by replacing $\mathbf{A}_{\mathrm{P}}, V_0$ with $\mathbf{A}^{\mathrm{LP}}, V_0^{\mathrm{LP}}$ in the Hamiltonian H_ω of (2.1). We choose the integer ℓ_{P} so that the unit cell of the large triangular lattice $\ell_{\mathrm{P}} \mathbf{L}^2$ contains the rectangular region Λ^{sys} of (2.3) on which the present finite Hall system is defined.

We note that the magnetic translations act on the random potential V_ω as the corresponding translation, and that the pair index does not depend on the location $\mathbf{a}$ of the flux [10, 11]. Therefore the pair index for the Hamiltonian H_ω^{LP} is an invariant function of the randomness under the lattice translations of the triangular lattice $\ell_P \mathbf{L}^2$. Further, since the index is measurable [10, 11] and integrable with respect to the random variables, Birkhoff's ergodic theorem implies that [21] the value of the pair index does not fluctuate in the sense that it takes an integer given by its mean for almost every random potentials. But the value of the integer may depend on the period ℓ_{P}.

As in [10, 11], the relation between the pair index, $\mathrm{Index}(P_{\mathrm{F}}, U_{\mathbf{a}}^\varepsilon P_{\mathrm{F}}(U_{\mathbf{a}}^\varepsilon)^*)$, and the Fredholm index, $\mathrm{Index}(P_{\mathrm{F}} U_{\mathbf{a}}^\varepsilon P_{\mathrm{F}})$, of the operator $P_{\mathrm{F}} U_{\mathbf{a}}^\varepsilon P_{\mathrm{F}}$ in range of P_{F} is given by

$$\mathrm{Index}(P_{\mathrm{F}} U_{\mathbf{a}}^\varepsilon P_{\mathrm{F}}) = -\mathrm{Index}(P_{\mathrm{F}}, U_{\mathbf{a}}^\varepsilon P_{\mathrm{F}}(U_{\mathbf{a}}^\varepsilon)^*). \tag{9.14}$$

Consider another unitary operator,

$$U_{\mathbf{a}} := \frac{x + iy - (a_1 + ia_2)}{|x + iy - (a_1 + ia_2)|}. \tag{9.15}$$

Clearly, one has $P_{\mathrm{F}} U_{\mathbf{a}} P_{\mathrm{F}} = P_{\mathrm{F}} U_{\mathbf{a}}^\varepsilon P_{\mathrm{F}} + P_{\mathrm{F}}(U_{\mathbf{a}} - U_{\mathbf{a}}^\varepsilon) P_{\mathrm{F}}$. Since the operator of the second term is compact, stability theory[9] of the Fredholm indices implies that $P_{\mathrm{F}} U_{\mathbf{a}} P_{\mathrm{F}}$ becomes a Fredholm operator, too, and that the index is invariant under the compact perturbation. Thus we have

$$\mathrm{Index}(P_{\mathrm{F}} U_{\mathbf{a}} P_{\mathrm{F}}) = \mathrm{Index}(P_{\mathrm{F}} U_{\mathbf{a}}^\varepsilon P_{\mathrm{F}}) = -\mathrm{Index}(P_{\mathrm{F}}, U_{\mathbf{a}}^\varepsilon P_{\mathrm{F}}(U_{\mathbf{a}}^\varepsilon)^*). \tag{9.16}$$

In consequence, the pair index does not depend on the parameters $\varepsilon_1, \varepsilon_2$.

Following [23], we obtain the expression (9.26) below with (9.27) for the pair index. The expression leads to the well-known Hall conductance formula [20] which is written in terms of the position operator of the electron.

To begin with, we note that

$$\begin{aligned}
&\mathrm{Tr}(P_{\mathrm{F}} - U_{\mathbf{a}}^\varepsilon P_{\mathrm{F}}(U_{\mathbf{a}}^\varepsilon)^*)^3 \\
&\quad = \sum_{\mathbf{u},\mathbf{v},\mathbf{w}} \mathrm{Tr}\, \chi_\varepsilon(\mathbf{u})(P_{\mathrm{F}} - U_{\mathbf{a}}^\varepsilon P_{\mathrm{F}}(U_{\mathbf{a}}^\varepsilon)^*)\chi_\varepsilon(\mathbf{v})(P_{\mathrm{F}} - U_{\mathbf{a}}^\varepsilon P_{\mathrm{F}}(U_{\mathbf{a}}^\varepsilon)^*)\chi_\varepsilon(\mathbf{w}) \\
&\qquad \times (P_{\mathrm{F}} - U_{\mathbf{a}}^\varepsilon P_{\mathrm{F}}(U_{\mathbf{a}}^\varepsilon)^*)\chi_\varepsilon(\mathbf{u}) \\
&\quad = \sum_{\mathbf{u},\mathbf{v},\mathbf{w}} t_{\mathbf{u},\mathbf{v}} t_{\mathbf{v},\mathbf{w}} t_{\mathbf{w},\mathbf{u}}\, \mathrm{Tr}\, \chi_\varepsilon(\mathbf{u}) P_{\mathrm{F}} \chi_\varepsilon(\mathbf{v}) P_{\mathrm{F}} \chi_\varepsilon(\mathbf{w}) P_{\mathrm{F}} \chi_\varepsilon(\mathbf{u}).
\end{aligned} \tag{9.17}$$

Since the index is independent of the location $\mathbf{a}$ of the flux [10, 11], one has

$$\begin{aligned}
&\mathrm{Index}(P_{\mathrm{F}} U_{\mathbf{a}} P_{\mathrm{F}}) \\
&\quad = -\frac{1}{\mathcal{V}_\ell} \sum_{\mathbf{a} \in \Lambda_\ell} \sum_{\mathbf{u},\mathbf{v},\mathbf{w}} t_{\mathbf{u},\mathbf{v}} t_{\mathbf{v},\mathbf{w}} t_{\mathbf{w},\mathbf{u}}\, \mathrm{Tr}\, \chi_\varepsilon(\mathbf{u}) P_{\mathrm{F}} \chi_\varepsilon(\mathbf{v}) P_{\mathrm{F}} \chi_\varepsilon(\mathbf{w}) P_{\mathrm{F}} \chi_\varepsilon(\mathbf{u}),
\end{aligned} \tag{9.18}$$

where $\Lambda_\ell = \varepsilon_1\{-\ell, -\ell+1, \ldots, \ell\} \times \varepsilon_2\{-\ell', -\ell'+1, \ldots, \ell'\} \subset \mathbf{Z}_\varepsilon^2$, and $\mathcal{V}_\ell = (2\ell+1) \times (2\ell'+1)$. We choose ℓ' so that the ratio ℓ'/ℓ is finite.

[9] See, for example, the book [44].

Lemma 9.2 *There exists a sequence $\{\ell_n\}_n$ such that for almost every ω, the index is written*

$$\mathrm{Index}(P_{\mathrm{F}}U_{\mathbf{a}}P_{\mathrm{F}}) = -\lim_{\ell_n\uparrow\infty}\frac{1}{\mathcal{V}_{\ell_n}}\sum_{\mathbf{u}\in\Lambda^*_{\ell_n}}\sum_{\mathbf{v},\mathbf{w}}\sum_{\mathbf{a}\in\mathbf{Z}^2_\varepsilon} t_{\mathbf{u},\mathbf{v}}t_{\mathbf{v},\mathbf{w}}t_{\mathbf{w},\mathbf{u}}S_{\mathbf{u},\mathbf{v},\mathbf{w},\mathbf{u}} \tag{9.19}$$

with

$$S_{\mathbf{u},\mathbf{v},\mathbf{w},\mathbf{u}} := \mathrm{Tr}\,\chi_\varepsilon(\mathbf{u})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{v})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{w})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{u}), \tag{9.20}$$

where the lattice Λ^*_ℓ is given by

$$\Lambda^*_\ell = \varepsilon_1\{-\ell+1/2, -\ell+3/2, \dots, \ell-1/2\} \\ \times \varepsilon_2\{-\ell'+1/2, -\ell'+3/2, \dots, \ell'-1/2\}. \tag{9.21}$$

The proof is given in Appendix G. Using Connes' area formula [45],

$$\sum_{\mathbf{a}\in\mathbf{Z}^2_\varepsilon} t_{\mathbf{u},\mathbf{v}}t_{\mathbf{v},\mathbf{w}}t_{\mathbf{w},\mathbf{u}} = \frac{2\pi i}{\varepsilon_1\varepsilon_2}(\mathbf{v}-\mathbf{u})\times(\mathbf{w}-\mathbf{u}), \tag{9.22}$$

the index of (9.19) is written

$$\begin{aligned}&\mathrm{Index}(P_{\mathrm{F}}U_{\mathbf{a}}P_{\mathrm{F}})\\ &\quad= -\lim_{|\Omega|\uparrow\infty}\frac{2\pi i}{|\Omega|}\sum_{\mathbf{u}\in\Lambda^*_{\ell_n}}\sum_{\mathbf{v},\mathbf{w}}(\mathbf{v}-\mathbf{u})\times(\mathbf{w}-\mathbf{u})\,\mathrm{Tr}\,\chi_\varepsilon(\mathbf{u})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{v})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{w})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{u}),\end{aligned} \tag{9.23}$$

where Ω is the $L_1\times L_2$ rectangular box centered at $\mathbf{r}=0$ with the sidelengths $L_1 = 2\ell_n\varepsilon_1$ and $L_2 = 2\ell'_n\varepsilon_2$. Note that

$$\begin{aligned}(\mathbf{v}-\mathbf{u})\times(\mathbf{w}-\mathbf{u}) &= (\mathbf{v}-\mathbf{w})\times(\mathbf{w}-\mathbf{u})\\ &= (v_2-w_2)(w_1-u_1)-(v_1-w_1)(w_2-u_2).\end{aligned} \tag{9.24}$$

Further we have

$$\begin{aligned}&(v_2-w_2)(w_1-u_1)\,\mathrm{Tr}\,\chi_\varepsilon(\mathbf{u})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{v})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{w})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{u})\\ &\quad= \mathrm{Tr}\,\chi_\varepsilon(\mathbf{u})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{v})[X_2^\varepsilon, P_{\mathrm{F}}]\chi_\varepsilon(\mathbf{w})[X_1^\varepsilon, P_{\mathrm{F}}]\chi_\varepsilon(\mathbf{u}),\end{aligned} \tag{9.25}$$

where the two-component function $(X_1^\varepsilon, X_2^\varepsilon)$ of $\mathbf{r}=(x,y)$ is given by $(X_1^\varepsilon, X_2^\varepsilon)=(u_1,u_2)$ for $\mathbf{r}$ in the $\varepsilon_1\times\varepsilon_2$ rectangular box $s_\varepsilon(\mathbf{u})$ centered at $\mathbf{u}=(u_1,u_2)$. From these observations, the index is written

$$\mathrm{Index}(P_{\mathrm{F}}U_{\mathbf{a}}P_{\mathrm{F}}) = \lim_{|\Omega|\uparrow\infty}\mathcal{I}^\varepsilon(P_{\mathrm{F}};\Omega,\ell_P) \tag{9.26}$$

for almost every ω, where we have written

$$\mathcal{I}^\varepsilon(P_{\mathrm{F}};\Omega,\ell_P) = \frac{2\pi i}{|\Omega|}\mathrm{Tr}\,\chi_\Omega P_{\mathrm{F}}[[P_{\mathrm{F}}, X_1^\varepsilon],[P_{\mathrm{F}}, X_2^\varepsilon]]\chi_\Omega. \tag{9.27}$$

From the proof of Lemma 9.2, we obtain

$$\mathbf{E}[|\mathrm{Index}(P_{\mathrm{F}}U_{\mathbf{a}}P_{\mathrm{F}}) - \mathcal{I}^\varepsilon(P_{\mathrm{F}};\Omega,\ell_P)|]\to 0 \quad \text{as } |\Omega|\uparrow\infty. \tag{9.28}$$

We also write

$$\mathcal{I}(P_{\mathrm{F}};\Omega,\ell_P):=\frac{2\pi i}{|\Omega|}\operatorname{Tr}\chi_\Omega P_{\mathrm{F}}[[P_{\mathrm{F}},X_1],[P_{\mathrm{F}},X_2]]\chi_\Omega, \tag{9.29}$$

where $(X_1,X_2)=(x,y)$. This right-hand side is nothing but the well-known form of the Hall conductance formula [20].

Lemma 9.3 *The following bound is valid*:

$$\mathbf{E}[|\mathcal{I}(P_{\mathrm{F}};\Omega,\ell_P)-\mathcal{I}^\varepsilon(P_{\mathrm{F}};\Omega,\ell_P)|]\le \mathrm{Const}\times|\varepsilon|, \tag{9.30}$$

where $|\varepsilon|=\sqrt{\varepsilon_1^2+\varepsilon_2^2}$, *and the positive constant in the right-hand side is independent of* Ω *and* ℓ_P.

Proof Note that

$$\begin{aligned}&|\operatorname{Tr}\chi_\Omega P_{\mathrm{F}}[P_{\mathrm{F}},X_1][P_{\mathrm{F}},X_2]\chi_\Omega-\operatorname{Tr}\chi_\Omega P_{\mathrm{F}}[P_{\mathrm{F}},X_1^\varepsilon][P_{\mathrm{F}},X_2^\varepsilon]\chi_\Omega|\\&\quad\le|\operatorname{Tr}\chi_\Omega P_{\mathrm{F}}[P_{\mathrm{F}},(X_1-X_1^\varepsilon)][P_{\mathrm{F}},X_2]\chi_\Omega|\\&\qquad+|\operatorname{Tr}\chi_\Omega P_{\mathrm{F}}[P_{\mathrm{F}},X_1^\varepsilon][P_{\mathrm{F}},(X_2-X_2^\varepsilon)]\chi_\Omega|,\end{aligned} \tag{9.31}$$

$$\begin{aligned}&[P_{\mathrm{F}},X_2]=[P_{\mathrm{F}},(y-y^b)]+[P_{\mathrm{F}},y^b],\\&[P_{\mathrm{F}},X_1^\varepsilon]=[P_{\mathrm{F}},(X_1^\varepsilon-x^b)]+[P_{\mathrm{F}},x^b].\end{aligned} \tag{9.32}$$

Therefore we can prove the statement of the theorem in the same way as in the proof of Lemma 8.3. □

Since we can apply Borel-Cantelli theorem as in the proof of Theorem 8.5 to this result, Lemma 9.3 yields that there exists a sequence $\{\varepsilon_n=(\varepsilon_{1,n},\varepsilon_{2,n})\}_n$ of $\varepsilon=\varepsilon_n$ satisfying $\varepsilon_{j,n}\to 0$ as $n\to\infty$, for $j=1,2$, such that

$$|\mathcal{I}(P_{\mathrm{F}};\Omega,\ell_P)-\mathcal{I}^{\varepsilon_n}(P_{\mathrm{F}};\Omega,\ell_P)|\to 0\quad \text{as } n\to\infty \tag{9.33}$$

almost surely for any fixed large Ω.

Let $\{\delta_n\}_{n=1}^\infty$ be a sequence of positive numbers δ_n satisfying $\delta_n\to 0$ as $n\to\infty$, and let $\{p_n\}_{n=1}^\infty$ be a sequence of positive numbers $p_n<1$ satisfying $\sum_n p_n<\infty$. Relying on Lemma 9.3, we choose $\varepsilon=\varepsilon_n=(\varepsilon_{1,n},\varepsilon_{2,n})$ for each n so that

$$\mathrm{Prob}[|\mathcal{I}(P_{\mathrm{F}};\Omega,\ell_P)-\mathcal{I}^{\varepsilon_n}(P_{\mathrm{F}};\Omega,\ell_P)|>\delta_n/2]\le p_n. \tag{9.34}$$

Further, for this $\varepsilon=\varepsilon_n$, we can choose a sufficiently large $\Omega=\Omega_n$ so that

$$\mathrm{Prob}[|\mathrm{Index}(P_{\mathrm{F}}U_{\mathbf{a}}P_{\mathrm{F}})-\mathcal{I}^{\varepsilon_n}(P_{\mathrm{F}};\Omega_n,\ell_P)|>\delta_n/2]\le p_n, \tag{9.35}$$

from the proof of Lemma 9.2. The application of Borel-Cantelli theorem yields that for almost every ω, there exists a number $n_0(\omega)$ such that for all $n\ge n_0(\omega)$, the following two inequalities are valid:

$$|\mathcal{I}(P_{\mathrm{F}};\Omega_n,\ell_P)-\mathcal{I}^{\varepsilon_n}(P_{\mathrm{F}};\Omega_n,\ell_P)|\le\delta_n/2 \tag{9.36}$$

and

$$|\mathrm{Index}(P_{\mathrm{F}}U_{\mathbf{a}}P_{\mathrm{F}}) - \mathcal{I}^{\varepsilon_n}(P_{\mathrm{F}}; \Omega_n, \ell_P)| \le \delta_n/2. \tag{9.37}$$

These two inequalities imply

$$\begin{aligned} &|\mathrm{Index}(P_{\mathrm{F}}U_{\mathbf{a}}P_{\mathrm{F}}) - \mathcal{I}(P_{\mathrm{F}}; \Omega_n, \ell_P)| \\ &\quad \le |\mathrm{Index}(P_{\mathrm{F}}U_{\mathbf{a}}P_{\mathrm{F}}) - \mathcal{I}^{\varepsilon_n}(P_{\mathrm{F}}; \Omega_n, \ell_P)| + |\mathcal{I}^{\varepsilon_n}(P_{\mathrm{F}}; \Omega_n, \ell_P) - \mathcal{I}(P_{\mathrm{F}}; \Omega_n, \ell_P)| \\ &\quad \le \delta_n/2 + \delta_n/2 = \delta_n. \end{aligned} \tag{9.38}$$

This result is summarized as the following index theorem:

Theorem 9.4 *For a fixed period ℓ_P and for almost every ω, there exists a sequence $\{\Omega_n\}_n$ of the regions $\Omega = \Omega_n$ going to $\mathbf{R}^2$ as $n \to \infty$ such that*

$$\mathrm{Index}(P_{\mathrm{F}}U_{\mathbf{a}}P_{\mathrm{F}}) = \lim_{\Omega_n \uparrow \mathbf{R}^2} \frac{2\pi i}{|\Omega_n|} \operatorname{Tr} \chi_{\Omega_n} P_{\mathrm{F}}[[P_{\mathrm{F}}, X_1], [P_{\mathrm{F}}, X_2]]\chi_{\Omega_n}. \tag{9.39}$$

For taking the infinite-volume limit $\Lambda \uparrow \mathbf{R}^2$, we want to take a sequence $\{\Lambda_n\}_n$ of the finite region $\Lambda = \Lambda_n$ of the present system so that the condition (8.4) is satisfied for $\Lambda = \Lambda_n \supset \Omega = \Omega_n$. Then we must take the limit $\ell_P \uparrow \infty$ together with the limit $\Omega_n \uparrow \mathbf{R}^2$ so that the unit cell of the lattice $\ell_P \mathbf{L}^2$ includes the region $\Lambda = \Lambda_n$ of the present system. In the following, we consider a sequence $\{\Lambda_n, \Omega_n, \ell_{\mathrm{P},n}\}_n$ which satisfies this requirement.

Since both of the key bounds in the proofs of Lemmas 9.2 and 9.3 do not depend on the period ℓ_{P}, we can take the limit $\ell_P \uparrow \infty$ together with the limit $\Omega_n \uparrow \mathbf{R}^2$ in the above argument for Theorem 9.4 so that the above requirement is satisfied. But both of $\mathrm{Index}(P_{\mathrm{F}}U_{\mathbf{a}}P_{\mathrm{F}})$ and $\mathcal{I}(P_{\mathrm{F}}; \Omega_n, \ell_P)$ may go to infinity as $\ell_P \uparrow \infty$. First let us prove that this case does not occur. From the argument of the proof of Lemma 8.3, one can easily show that the expectation value $\mathbf{E}[|\mathcal{I}(P_{\mathrm{F}}; \Omega, \ell_P)|]$ is bounded uniformly in Ω and ℓ_{P}. Combining this with Fatou's lemma, we have

$$\mathbf{E}\Big[\liminf_{\Omega \uparrow \mathbf{R}^2, \ell_{\mathrm{P}} \uparrow \infty} |\mathcal{I}(P_{\mathrm{F}}; \Omega, \ell_P)|\Big] \le \liminf_{\Omega \uparrow \mathbf{R}^2, \ell_{\mathrm{P}} \uparrow \infty} \mathbf{E}[|\mathcal{I}(P_{\mathrm{F}}; \Omega, \ell_P)|] < \infty. \tag{9.40}$$

This implies that for almost every ω, there exists a sequence $\{\Omega_n(\omega), \ell_{\mathrm{P},n}(\omega)\}_n$ of the pair $\{\Omega, \ell_{\mathrm{P}}\}$ such that $\{\Omega_n(\omega)\}$ is a subsequence of the sequence $\{\Omega_n\}$ of Theorem 9.4, and that $\lim_{n\uparrow\infty} \mathcal{I}(P_{\mathrm{F}}; \Omega_n(\omega), \ell_{\mathrm{P},n}(\omega))$ exists. Here we should stress that the sequence $\{\Omega_n(\omega), \ell_{\mathrm{P},n}(\omega)\}_n$ may depend on the random event ω. On the other hand, the inequality (9.38) holds for a large pair $\{\Omega, \ell_P\} = \{\Omega_n(\omega), \ell_{\mathrm{P},n}(\omega)\}$. These observations imply that for a fixed ω, the index $\mathrm{Index}(P_{\mathrm{F}}U_{\mathbf{a}}P_{\mathrm{F}})$ converges to an integer as $n \uparrow \infty$, too. But, since the index does not depend on ω as mentioned above, we can write $\{\ell_{\mathrm{P},n}\}_n$ for the sequence $\{\ell_{\mathrm{P},n}(\omega)\}_n$ by dropping the ω dependence, and obtain the result that the following limit exists and is constant for almost every ω:

$$\mathrm{Index}_\infty(P_{\mathrm{F}}U_{\mathbf{a}}P_{\mathrm{F}}) := \lim_{\ell_{\mathrm{P},n} \uparrow \infty} \mathrm{Index}(P_{\mathrm{F}}U_{\mathbf{a}}P_{\mathrm{F}}). \tag{9.41}$$

Newly we choose $\{\Omega, \ell_P\} = \{\Omega_n, \ell_{\mathrm{P},n}\}$ in the inequality (9.38). Then, since the index converges to the integer as $n \uparrow \infty$, we obtain

Theorem 9.5 *For almost every ω, there exists a sequence $\{\Omega_n, \ell_{\mathrm{P},n}\}_n$ of the pair $\{\Omega = \Omega_n, \ell_{\mathrm{P}} = \ell_{\mathrm{P},n}\}$ such that the following relation holds*:

$$\mathrm{Index}_\infty(P_{\mathrm{F}} U_{\mathbf{a}} P_{\mathrm{F}}) = \lim_{\Omega_n \uparrow \mathbf{R}^2, \ell_{\mathrm{P},n} \uparrow \infty} \frac{2\pi i}{|\Omega_n|} \mathrm{Tr}\, \chi_{\Omega_n} P_{\mathrm{F}}[[P_{\mathrm{F}}, X_1], [P_{\mathrm{F}}, X_2]] \chi_{\Omega_n}. \tag{9.42}$$

Theorem 9.6 *There exists a subsequence $\{\Omega_n, \ell_{\mathrm{P},n}\}_n$ of the sequence of the preceding Theorem 9.5 such that*

$$\begin{aligned}\mathrm{Index}_\infty(P_{\mathrm{F}} U_{\mathbf{a}} P_{\mathrm{F}}) &= \lim_{\Omega_n \uparrow \mathbf{R}^2, \ell_{\mathrm{P},n} \uparrow \infty} \frac{2\pi i}{|\Omega_n|} \mathrm{Tr}\, \chi_{\Omega_n} P_{\mathrm{F}}[[P_{\mathrm{F}}, X_1], [P_{\mathrm{F}}, X_2]] \chi_{\Omega_n} \\ &= \lim_{\Omega_n \uparrow \mathbf{R}^2} \frac{2\pi i}{|\Omega_n|} \mathrm{Tr}\, \chi_{\Omega_n} P_{\mathrm{F},\Lambda_n}[[P_{\mathrm{F},\Lambda_n}, x], [P_{\mathrm{F},\Lambda_n}, y]] \chi_{\Omega_n}\end{aligned} \tag{9.43}$$

with probability one. Here we take $\Lambda = \Lambda_n$ and $\ell_{\mathrm{P}} = \ell_{\mathrm{P},n}$ so that the region $\Lambda = \Lambda_n$ of the present system satisfies the condition (8.4) for each $\Omega = \Omega_n$, and the unit cell of the lattice $\ell_{\mathrm{P},n}\mathbf{L}^2$ includes the region Λ_n.

Theorem 9.6 follows from the following lemma:

Lemma 9.7 *Under the same assumption as in Theorem 9.6, we have*

$$\mathbf{E}[|\mathcal{I}(P_{\mathrm{F}}; \Omega_n, \ell_{\mathrm{P},n}) - \mathcal{I}(P_{\mathrm{F},\Lambda_n}; \Omega_n)|] \to 0 \quad \textit{as } n \to \infty. \tag{9.44}$$

The proof is given in Appendix H.

Theorem 9.8 *There exists a sequence $\{L_n\}_n$ of the system sizes $L = L_n$ such that the Hall conductance σ_{xy} in the infinite volume limit exists and is quantized to an integer as*

$$\sigma_{xy} = \frac{e^2}{h} \mathrm{Index}_\infty(P_{\mathrm{F}} U_{\mathbf{a}} P_{\mathrm{F}}) \tag{9.45}$$

for almost every ω.

Proof From Theorems 9.5 and 9.6, there exists a sequence of $\{\Lambda_n, \Omega_n, \ell_{\mathrm{P},n}\}_n$ of the triplet $\{\Lambda, \Omega, \ell_{\mathrm{P}}\}$ such that the following three conditions are satisfied: (i) the condition (8.4) is satisfied for $\Lambda = \Lambda_n \supset \Omega = \Omega_n$, (ii) the unit cell with the period $\ell_{\mathrm{P},n}$ includes the region $\Lambda = \Lambda_n$ of the system with the linear size $L = L_n'$, and (iii) for almost every ω, the following formula holds:

$$\mathrm{Index}_\infty(P_{\mathrm{F}} U_{\mathbf{a}} P_{\mathrm{F}}) = \lim_{n \uparrow \infty} \frac{2\pi i}{|\Omega_n|} \mathrm{Tr}\, \chi_{\Omega_n} P_{\mathrm{F},\Lambda_n}[[P_{\mathrm{F},\Lambda_n}, x], [P_{\mathrm{F},\Lambda_n}, y]] \chi_{\Omega_n}. \tag{9.46}$$

Take a subsequence $\{L_n\}_n$ of $\{L_n'\}_n$ so that the sequence $\{L_n\}_n$ of the system sizes satisfies the two conditions of (8.34) in the proof of Lemma 8.5. Then, for almost every ω, the contribution $\sigma_{xy}^{\mathrm{out}}$ of the Hall conductance from the boundary region is vanishing as $n \uparrow \infty$, and the correction of the Hall conductance $\sigma_{xy}^{\mathrm{in}}$ of (8.15) for the bulk region is also vanishing in this limit. Combining this with (8.15) yields the desired result. □

10 Constancy of the Hall Conductance—Homotopy Argument

In this section, we prove that the Hall conductance σ_{xy} is constant as long as both the strengths of the potentials and the Fermi energy vary in the localization regime.

10.1 Changing the Strengths of the Potentials

Consider first changing the strengths of the potentials V_0^{LP}, $\mathbf{A}^{\mathrm{LP}}$, V_ω in the Hamiltonian H_ω^{LP} of (9.13) on the whole plane $\mathbf{R}^2$. In order to prove constancy of the Hall conductance, we extend the homotopy argument of [22] for lattice models to continuous models by relying on the fractional moment bound [32] for the resolvent. As a byproduct, we prove that the quantized value of the Hall conductance is independent of the period ℓ_{P} of the potentials V_0^{LP}, $\mathbf{A}^{\mathrm{LP}}$. See Theorem 10.1 below.

Since all the cases can be handled in the same way, we consider only the case where the strength of the vector potential $\mathbf{A}_{\mathrm{P}}$ varies. We denote by P_{F}' the Fermi sea projection for the Hamiltonian H_ω' with the vector potential $\mathbf{A}_{\mathrm{P}}' = \mathbf{A}_{\mathrm{P}} + \delta\mathbf{A}_{\mathrm{P}}$ with a small change $\||\delta\mathbf{A}_{\mathrm{P}}|\|_\infty$. Since the index does not depend on ω, we have

$$\begin{aligned}
&|\mathrm{Index}(P_{\mathrm{F}}'U_{\mathbf{a}}P_{\mathrm{F}}') - \mathrm{Index}(P_{\mathrm{F}}U_{\mathbf{a}}P_{\mathrm{F}})|\\
&\quad = \mathbf{E}[|\mathrm{Index}(P_{\mathrm{F}}'U_{\mathbf{a}}P_{\mathrm{F}}') - \mathrm{Index}(P_{\mathrm{F}}U_{\mathbf{a}}P_{\mathrm{F}})|]\\
&\quad \le \mathbf{E}[|\mathrm{Index}(P_{\mathrm{F}}'U_{\mathbf{a}}P_{\mathrm{F}}') - \mathcal{I}^{\varepsilon}(P_{\mathrm{F}}'; \varOmega, \ell_P)|] + \mathbf{E}[|\mathcal{I}^{\varepsilon}(P_{\mathrm{F}}'; \varOmega, \ell_P) - \mathcal{I}^{\varepsilon}(P_{\mathrm{F}}; \varOmega, \ell_P)|]\\
&\qquad + \mathbf{E}[|\mathcal{I}^{\varepsilon}(P_{\mathrm{F}}; \varOmega, \ell_P) - \mathrm{Index}(P_{\mathrm{F}}U_{\mathbf{a}}P_{\mathrm{F}})|]. \qquad (10.1)
\end{aligned}$$

From (9.28), the first and the third terms in the right-hand side become small for a large $\varOmega$. Therefore it is sufficient to show that the second term become small for a small change $\||\delta\mathbf{A}_{\mathrm{P}}|\|_\infty$ of the vector potential.

Relying on the expression given by the right-hand side of (9.23), let us estimate the difference,

$$\begin{aligned}
&\mathrm{Tr}\,\chi_\varepsilon(\mathbf{u})P_{\mathrm{F}}'\chi_\varepsilon(\mathbf{v})P_{\mathrm{F}}'\chi_\varepsilon(\mathbf{w})P_{\mathrm{F}}'\chi_\varepsilon(\mathbf{u}) - \mathrm{Tr}\,\chi_\varepsilon(\mathbf{u})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{v})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{w})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{u})\\
&\quad = \mathrm{Tr}\,\chi_\varepsilon(\mathbf{u})\Delta P_{\mathrm{F}}\chi_\varepsilon(\mathbf{v})P_{\mathrm{F}}'\chi_\varepsilon(\mathbf{w})P_{\mathrm{F}}'\chi_\varepsilon(\mathbf{u}) + \mathrm{Tr}\,\chi_\varepsilon(\mathbf{u})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{v})\Delta P_{\mathrm{F}}\chi_\varepsilon(\mathbf{w})P_{\mathrm{F}}'\chi_\varepsilon(\mathbf{u})\\
&\qquad + \mathrm{Tr}\,\chi_\varepsilon(\mathbf{u})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{v})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{w})\Delta P_{\mathrm{F}}\chi_\varepsilon(\mathbf{u}), \qquad (10.2)
\end{aligned}$$

where $\Delta P_{\mathrm{F}} = P_{\mathrm{F}}' - P_{\mathrm{F}}$. The first term in the right-hand side is estimated as

$$\begin{aligned}
&|\,\mathrm{Tr}\,\chi_\varepsilon(\mathbf{u})\Delta P_{\mathrm{F}}\chi_\varepsilon(\mathbf{v})P_{\mathrm{F}}'\chi_\varepsilon(\mathbf{w})P_{\mathrm{F}}'\chi_\varepsilon(\mathbf{u})|\\
&\quad \le \sqrt{\mathrm{Tr}\,\chi_\varepsilon(\mathbf{u})\Delta P_{\mathrm{F}}\chi_\varepsilon(\mathbf{v})P_{\mathrm{F}}'\chi_\varepsilon(\mathbf{v})\Delta P_{\mathrm{F}}\chi_\varepsilon(\mathbf{u})}\sqrt{\mathrm{Tr}\,\chi_\varepsilon(\mathbf{u})P_{\mathrm{F}}'\chi_\varepsilon(\mathbf{w})P_{\mathrm{F}}'\chi_\varepsilon(\mathbf{w})P_{\mathrm{F}}'\chi_\varepsilon(\mathbf{u})}\\
&\quad \le \mathrm{Const} \times \|\chi_\varepsilon(\mathbf{u})\Delta P_{\mathrm{F}}\chi_\varepsilon(\mathbf{v})\|\|\chi_\varepsilon(\mathbf{u})P_{\mathrm{F}}'\chi_\varepsilon(\mathbf{w})\| \qquad (10.3)
\end{aligned}$$

by using Schwarz's inequality and the bound (8.27). Similarly, the second term is estimated as

$$\begin{aligned}
&|\,\mathrm{Tr}\,\chi_\varepsilon(\mathbf{u})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{v})\Delta P_{\mathrm{F}}\chi_\varepsilon(\mathbf{w})P_{\mathrm{F}}'\chi_\varepsilon(\mathbf{u})|\\
&\quad \le \sqrt{\mathrm{Tr}\,\chi_\varepsilon(\mathbf{u})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{u})}\sqrt{\mathrm{Tr}\,\chi_\varepsilon(\mathbf{u})P_{\mathrm{F}}'\chi_\varepsilon(\mathbf{w})\Delta P_{\mathrm{F}}\chi_\varepsilon(\mathbf{v})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{v})\Delta P_{\mathrm{F}}\chi_\varepsilon(\mathbf{w})P_{\mathrm{F}}'\chi_\varepsilon(\mathbf{u})}\\
&\quad \le \mathrm{Const} \times \|\chi_\varepsilon(\mathbf{u})P_{\mathrm{F}}'\chi_\varepsilon(\mathbf{w})\Delta P_{\mathrm{F}}\chi_\varepsilon(\mathbf{v})\|\\
&\quad \le \mathrm{Const} \times \|\chi_\varepsilon(\mathbf{u})P_{\mathrm{F}}'\chi_\varepsilon(\mathbf{w})\|\|\chi_\varepsilon(\mathbf{w})\Delta P_{\mathrm{F}}\chi_\varepsilon(\mathbf{v})\|. \qquad (10.4)
\end{aligned}$$

The third term can be handled in the same way as for the first term.

The contour representation of the Fermi sea projection yields

$$\|\chi_\varepsilon(\mathbf{u})[P_{\mathrm{F}}' - P_{\mathrm{F}}]\chi_\varepsilon(\mathbf{v})\| \le I_{\mathbf{u},\mathbf{v}}^{(1)} + I_{\mathbf{u},\mathbf{v}}^{(2)} + I_{\mathbf{u},\mathbf{v}}^{(+)} + I_{\mathbf{u},\mathbf{v}}^{(-)} \tag{10.5}$$

with

$$I_{\mathbf{u},\mathbf{v}}^{(1)} = \frac{1}{2\pi}\int_{y_-}^{y_+} dy \|\chi_\varepsilon(\mathbf{u})[R'(E_{\mathrm{F}} + iy) - R(E_{\mathrm{F}} + iy)]\chi_\varepsilon(\mathbf{v})\|, \tag{10.6}$$

$$I_{\mathbf{u},\mathbf{v}}^{(2)} = \frac{1}{2\pi}\int_{y_-}^{y_+} dy \|\chi_\varepsilon(\mathbf{u})[R'(E_0 + iy) - R(E_0 + iy)]\chi_\varepsilon(\mathbf{v})\| \tag{10.7}$$

and

$$I_{\mathbf{u},\mathbf{v}}^{(\pm)} = \frac{1}{2\pi}\int_{E_0}^{E_{\mathrm{F}}} dE \|\chi_\varepsilon(\mathbf{u})[R'(E + iy_\pm) - R(E + iy_\pm)]\chi_\varepsilon(\mathbf{v})\|, \tag{10.8}$$

where $R'(z) = (z - H_\omega')^{-1}$ and $R(z) = (z - H_\omega)^{-1}$.

First let us estimate the last three integrals except for $I_{\mathbf{u},\mathbf{v}}^{(1)}$. Note that $R'(z) - R(z) = -R'(z)\delta H_\omega R(z)$, where

$$\delta H_\omega = \frac{e}{2m_e}[\delta\mathbf{A}\cdot(\mathbf{p} + e\mathbf{A}) + (\mathbf{p} + e\mathbf{A})\cdot\delta\mathbf{A}] + \frac{e^2}{2m_e}|\delta\mathbf{A}|^2. \tag{10.9}$$

Since all the contributions in the perturbation δH_ω can be handled in the same way, we consider only

$$\begin{aligned}
&\chi_\varepsilon(\mathbf{u})R'(z)\delta A_s(p_s + eA_s)R(z)\chi_\varepsilon(\mathbf{v}) \\
&\quad = \sum_{\mathbf{u}'} \chi_\varepsilon(\mathbf{u})R'(z)\tilde{\chi}_b(\mathbf{u}')\delta A_s(p_s + eA_s)\chi_b^\delta(\mathbf{u}')R(z)\chi_\varepsilon(\mathbf{v})
\end{aligned} \tag{10.10}$$

as a typical one. Here $\{\chi_b^\delta(\mathbf{u})\}_{\mathbf{u}}$ is the partition of unity which is given in the proof of Lemma H.1 in Appendix H, and $\tilde{\chi}_b(\mathbf{u})$ is the characteristic function of the support of $\chi_b^\delta(\mathbf{u})$. The norm is estimated as

$$\begin{aligned}
&\|\chi_\varepsilon(\mathbf{u})R'(z)\delta A_s(p_s + eA_s)R(z)\chi_\varepsilon(\mathbf{v})\| \\
&\quad \le \|\delta A_s\|_\infty \sum_{\mathbf{u}'} \|\chi_\varepsilon(\mathbf{u})R'(z)\tilde{\chi}_b(\mathbf{u}')\| \|(p_s + eA_s)\chi_b^\delta(\mathbf{u}')R(z)\chi_\varepsilon(\mathbf{v})\| \\
&\quad \le \mathrm{Const} \times \|\delta A_s\|_\infty \sum_{\mathbf{u}'} \|\chi_\varepsilon(\mathbf{u})R'(z)\tilde{\chi}_b(\mathbf{u}')\| \\
&\qquad \times [\|\tilde{\chi}_b(\mathbf{u}')R(z)\chi_\varepsilon(\mathbf{v})\| + \mathrm{Const} \times \|\tilde{\chi}_b(\mathbf{u}')R(z)\chi_\varepsilon(\mathbf{v})\|^{1/2}],
\end{aligned} \tag{10.11}$$

where we have used Lemma H.1 for getting the second inequality. Since $\mathrm{dist}(\sigma(H_\omega), z) > 0$ in the present situation, all the norms about the resolvent $R(z)$ decay exponentially at the large distance. Therefore we obtain

$$I_{\mathbf{u},\mathbf{v}}^{(\sharp)} \le \mathrm{Const} \times \||\delta\mathbf{A}_{\mathrm{P}}|\|_\infty \exp[-\mu'|\mathbf{u} - \mathbf{v}|] \tag{10.12}$$

with a positive constant μ', where $\sharp = 2, \pm$. Consider the contribution from (10.3) because the rest can be treated in the same way. The corresponding contribution is estimated by

$$\begin{aligned}&\frac{\||\delta \mathbf{A}_{\mathrm{P}}|\|_\infty}{|\Omega|}\sum_{\mathbf{u}\in\Lambda_\ell^*}\sum_{\mathbf{v},\mathbf{w}}|\mathbf{u}-\mathbf{v}||\mathbf{u}-\mathbf{w}|e^{-\mu'|\mathbf{u}-\mathbf{v}|}\mathbf{E}[\|\chi_\varepsilon(\mathbf{u})P_{\mathrm{F}}'\chi_\varepsilon(\mathbf{w})\|]\\&\quad\le \mathrm{Const}\times\||\delta\mathbf{A}_{\mathrm{P}}|\|_\infty,\end{aligned}\tag{10.13}$$

where we have used the decay bound (7.17) for the Fermi sea projection.

Let $s \in (0, 1/3)$. The rest of the integrals is written

$$\begin{aligned}I_{\mathbf{u},\mathbf{v}}^{(1)} &= \frac{1}{2\pi}\int_{y_-}^{y_+} dy\|\chi_\varepsilon(\mathbf{u})[R'(E_{\mathrm{F}}+iy)-R(E_{\mathrm{F}}+iy)]\chi_\varepsilon(\mathbf{v})\|^{s/3}\\&\quad\times\|\chi_\varepsilon(\mathbf{u})[R'(E_{\mathrm{F}}+iy)-R(E_{\mathrm{F}}+iy)]\chi_\varepsilon(\mathbf{v})\|^{1-s/3}\\&\le\frac{1}{\pi}\int_{y_-}^{y_+}dy\|\chi_\varepsilon(\mathbf{u})[R'(E_{\mathrm{F}}+iy)-R(E_{\mathrm{F}}+iy)]\chi_\varepsilon(\mathbf{v})\|^{s/3}|y|^{s/3-1},\end{aligned}\tag{10.14}$$

where we have used the inequality $(\sum_j a_j)^s \le \sum_j a_j^s$ for $s\in(0,1)$ and $a_j \ge 0$, and the inequality $\|R^\sharp(E_{\mathrm{F}}+iy)\|\le|y|^{-1}$ for $R^\sharp = R', R$. For the norm of the operator in the integrand, the contribution from the term (10.10) can be estimated as

$$\begin{aligned}&\|\chi_\varepsilon(\mathbf{u})R'(E_{\mathrm{F}}+iy)\delta A_s(p_s+eA_s)R(E_{\mathrm{F}}+iy)\chi_\varepsilon(\mathbf{v})\|^{s/3}\\&\quad\le\mathrm{Const}\times\|\delta A_s\|_\infty\sum_{\mathbf{u}'}\{\|\chi_\varepsilon(\mathbf{u})R'(E_{\mathrm{F}}+iy)\tilde\chi_b(\mathbf{u}')\|^{s/3}\|\tilde\chi_b(\mathbf{u}')R(E_{\mathrm{F}}+iy)\chi_\varepsilon(\mathbf{v})\|^{s/3}\\&\qquad+\mathrm{Const}\times\|\chi_\varepsilon(\mathbf{u})R'(E_{\mathrm{F}}+iy)\tilde\chi_b(\mathbf{u}')\|^{s/3}\|\tilde\chi_b(\mathbf{u}')R(E_{\mathrm{F}}+iy)\chi_\varepsilon(\mathbf{v})\|^{s/6}\},\end{aligned}\tag{10.15}$$

where we have used Lemma H.1 for getting the inequality.

Consider the contribution from the first term in the summand in the right-hand side of (10.15) because the second term can be handled in the same way. The corresponding contribution from (10.3) is estimated by

$$\begin{aligned}&\frac{\||\delta\mathbf{A}_{\mathrm{P}}|\|_\infty}{|\Omega|}\sum_{\mathbf{u}\in\Lambda_\ell^*}\sum_{\mathbf{v},\mathbf{w}}\sum_{\mathbf{u}'}|\mathbf{u}-\mathbf{v}||\mathbf{u}-\mathbf{w}|\mathbf{E}\int_{y_-}^{y_+}dy|y|^{s/3-1}\\&\quad\times\|\chi_\varepsilon(\mathbf{u})R'(E_{\mathrm{F}}+iy)\tilde\chi_b(\mathbf{u}')\|^{s/3}\|\tilde\chi_b(\mathbf{u}')R(E_{\mathrm{F}}+iy)\chi_\varepsilon(\mathbf{v})\|^{s/3}\\&\quad\times\|\chi_\varepsilon(\mathbf{u})P_{\mathrm{F}}'\chi_\varepsilon(\mathbf{w})\|.\end{aligned}\tag{10.16}$$

Using Hölder's inequality, we have

$$\begin{aligned}&\mathbf{E}[\|\chi_\varepsilon(\mathbf{u})R'(E_{\mathrm{F}}+iy)\tilde\chi_b(\mathbf{u}')\|^{s/3}\|\tilde\chi_b(\mathbf{u}')R(E_{\mathrm{F}}+iy)\chi_\varepsilon(\mathbf{v})\|^{s/3}\|\chi_\varepsilon(\mathbf{u})P_{\mathrm{F}}'\chi_\varepsilon(\mathbf{w})\|]\\&\quad\le\{\mathbf{E}[\|\chi_\varepsilon(\mathbf{u})R'(E_{\mathrm{F}}+iy)\tilde\chi_b(\mathbf{u}')\|^s]\}^{1/3}\{\mathbf{E}[\|\tilde\chi_b(\mathbf{u}')R(E_{\mathrm{F}}+iy)\chi_\varepsilon(\mathbf{v})\|^s]\}^{1/3}\\&\qquad\times\{\mathbf{E}[\|\chi_\varepsilon(\mathbf{u})P_{\mathrm{F}}'\chi_\varepsilon(\mathbf{w})\|^3]\}^{1/3}\\&\quad\le\mathrm{Const}\times e^{-\mu|\mathbf{u}-\mathbf{u}'|/3}e^{-\mu|\mathbf{u}'-\mathbf{v}|/3}e^{-\mu|\mathbf{u}-\mathbf{w}|/3},\end{aligned}\tag{10.17}$$

where we have used the decay bounds (7.1), (7.17) and $\|\chi_\varepsilon(\mathbf{u})P_{\mathrm{F}}'\chi_\varepsilon(\mathbf{w})\|\le 1$. Relying on Fatou's lemma and Fubini-Tonelli theorem, and substituting the bound (10.17) into (10.16), we can obtain the desired result.

Thus the index is constant as long as the strengths of the potentials vary in the localization regime. In order to describe our statement more precisely, we recall the definitions of the lower and upper localization regimes which are given by the intervals (5.36) and (5.37), respectively, in the case of $\mathbf{A}_{\mathrm{P}} = 0$. In the case of $\mathbf{A}_{\mathrm{P}} \neq 0$, the corresponding condition is given by (5.47). See also Sect. 11. We continuously change the strength of all the potentials, V_0, V_ω and $\mathbf{A}_{\mathrm{P}}$, starting from the special point, $\|V_0\|_\infty = \|V_\omega\|_\infty = 0$ and $\||\mathbf{A}_{\mathrm{P}}|\|_\infty = 0$, for a fixed Fermi energy E_{F}. Then, if the Fermi energy E_{F} is staying the lower or upper localization regime, the index is equal to the special case that all the potentials are vanishing. This result also implies that the index does not depend on the period ℓ_P of the potentials. In consequence, Theorem 9.8 is refined as

Theorem 10.1 *Suppose that the Fermi energy E_{F} lies in the localization regime around the n-th Landau energy $\mathcal{E}_{n-1} = (n - 1/2)\hbar\omega_c$. There exists a sequence $\{L_{x,n}, L_{y,n}\}_n$ of the system sizes such that the Hall conductance σ_{xy} in the infinite volume limit exists and is quantized to an integer as*

$$\sigma_{xy} = -\frac{e^2}{h} \times \begin{cases} n & \textit{for the upper localization regime}, \\ n-1 & \textit{for the lower localization regime} \end{cases} \tag{10.18}$$

with probability one.

Remark

1. When the strength of one of the potentials becomes sufficiently large for a fixed strength of the magnetic field, the localization regimes become empty in our definition. Thus we need the condition that the strengths of the potentials are weak, compared to the strength of the magnetic field.
2. The number of the states in a localization regime will be proved to be of bulk order for the weak potentials, compared to the strength of the magnetic field in Sect. 11.
3. We do not require any assumption on the tails $\lambda \in [\lambda_{\min}, -\lambda_-] \cup [\lambda_+, \lambda_{\max}]$ of the coupling constant of the random potential V_ω. Therefore we allow the possibility that the spectral gap between two neighboring disordered-broadened Landau bands vanishes owing to the tails of the random potential.

10.2 Changing the Fermi Level

Next let us prove the constancy of the Hall conductance for changing the Fermi level E_{F}.

The Hall conductance σ_{xy} of (8.2) is written

$$\sigma_{xy} = -\frac{i\hbar e^2}{L_x L_y} \sum_{m,n:E_m<E_F<E_n} \left[\frac{(\varphi_m, v_x\varphi_n)(\varphi_n, v_y\varphi_m)}{(E_m - E_n)^2} - (x \leftrightarrow y)\right] \tag{10.19}$$

in terms of the eigenvector φ_n of the single electron Hamiltonian H_ω with the eigenvalue E_n, $n = 1, 2, \ldots$, on the box Λ^{sys}. We take the energy eigenvalues E_n in increasing order, repeated according to multiplicity.

Consider changing the number of the electrons below the Fermi level from N to N' in the localization regime. Without loss of generality, we can assume $N' > N$. We denote by E_{F} and E'_{F} the corresponding two Fermi energies for N and N' electrons, respectively. The sum in the right-hand side of (10.19) for N' electrons is written as

$$\begin{aligned}
&\sum_{m,n:E_m<E_F'<E_n}\left[\frac{(\varphi_m,v_x\varphi_n)(\varphi_n,v_y\varphi_m)}{(E_m-E_n)^2}-(x\leftrightarrow y)\right]\\
&=\sum_{m\le N}\sum_{n=N'+1}^{\infty}\left[\frac{(\varphi_m,v_x\varphi_n)(\varphi_n,v_y\varphi_m)}{(E_m-E_n)^2}-(x\leftrightarrow y)\right]\\
&\quad+\sum_{m=N+1}^{N'}\sum_{n=N'+1}^{\infty}\left[\frac{(\varphi_m,v_x\varphi_n)(\varphi_n,v_y\varphi_m)}{(E_m-E_n)^2}-(x\leftrightarrow y)\right]\\
&=\sum_{m\le N}\sum_{n=N+1}^{\infty}\left[\frac{(\varphi_m,v_x\varphi_n)(\varphi_n,v_y\varphi_m)}{(E_m-E_n)^2}-(x\leftrightarrow y)\right]\\
&\quad-\sum_{m\le N}\sum_{n=N+1}^{N'}\left[\frac{(\varphi_m,v_x\varphi_n)(\varphi_n,v_y\varphi_m)}{(E_m-E_n)^2}-(x\leftrightarrow y)\right]\\
&\quad+\sum_{m=N+1}^{N'}\sum_{n=N'+1}^{\infty}\left[\frac{(\varphi_m,v_x\varphi_n)(\varphi_n,v_y\varphi_m)}{(E_m-E_n)^2}-(x\leftrightarrow y)\right].
\end{aligned}\tag{10.20}$$

The first double sum in the right-hand side of the second equality leads the Hall conductance σ_{xy} for N electrons. Therefore it is sufficient to estimate the other two double sums. These two sums are compactly written as

$$\sum_{m=N+1}^{N'}\sum_{n\le N\ \text{and}\ n>N'}\left[\frac{(\varphi_m,v_x\varphi_n)(\varphi_n,v_y\varphi_m)}{(E_m-E_n)^2}-(x\leftrightarrow y)\right].\tag{10.21}$$

In consequence, the difference between the two Hall conductances for N and N' electrons is written

$$\Delta\sigma_{xy}^{\rm loc}=-\frac{i\hbar e^2}{L_xL_y}\,{\rm Tr}\,\Delta P_\Lambda^{\rm loc}[P_{x,\Lambda},P_{y,\Lambda}]\Delta P_\Lambda^{\rm loc},\tag{10.22}$$

where $\Delta P_\Lambda^{\rm loc}$ is the spectral projection onto the localization regime, and

$$P_{s,\Lambda}=\frac{1}{2\pi i}\int_\gamma dz R_\Lambda(z)v_sR_\Lambda(z),\tag{10.23}$$

with the resolvent $R_\Lambda(z)=(z-H_{\omega,\Lambda})^{-1}$ for the present Hamiltonian $H_{\omega,\Lambda}$ on the box $\Lambda=\Lambda^{\rm sys}$. Here the closed path γ encircles the energy eigenvalues of the "localized" states. In the same way as in Lemma 8.2 and (8.32), we have

$$\Delta\sigma_{xy}^{\rm loc}=\frac{e^2}{h}\mathcal{I}^{\rm loc}(\Delta P_\Lambda^{\rm loc})+\delta(L)+\mathcal{O}(\exp[-\mu'L^{2\kappa/3}])\tag{10.24}$$

with probability larger than $(1-{\rm Const}\times L^{-2[\kappa(\xi+2)-3]/3})$, where

$$\mathcal{I}^{\rm loc}(\Delta P_\Lambda^{\rm loc})=\frac{2\pi i}{|\Omega|}\,{\rm Tr}\,\chi_\Omega\Delta P_\Lambda^{\rm loc}[[\Delta P_\Lambda^{\rm loc},x],[\Delta P_\Lambda^{\rm loc},y]]\chi_\Omega,\tag{10.25}$$

and $\delta(L)$ is the correction which comes from the boundary region $\Lambda\backslash\Omega$. In the same way as in the proof of Lemma 8.5, we can show that $\mathbf{E}[\delta(L)] \to 0$ as $L \to \infty$.

Lemma 10.2 *The following bound is valid*:

$$\mathbf{E}|\mathcal{I}^{\mathrm{loc}}(\Delta P_\Lambda^{\mathrm{loc}})| \le \mathrm{Const} \times \Delta E^{1/2}, \tag{10.26}$$

where $\Delta E = E_{\mathrm{F}}' - E_{\mathrm{F}}$.

Proof Using Schwarz's inequality, one has

$$\mathbf{E}|\mathcal{I}^{\mathrm{loc}}(\Delta P_\Lambda^{\mathrm{loc}})| \le \frac{2\pi}{|\Omega|}\sqrt{\mathbf{E}[\mathrm{Tr}\,\chi_\Omega \Delta P_\Lambda^{\mathrm{loc}} \chi_\Omega] \cdot \mathbf{E}[\mathrm{Tr}\,\chi_\Omega A^* \Delta P_\Lambda^{\mathrm{loc}} A \chi_\Omega]}, \tag{10.27}$$

where we have written $A = [[\Delta P_\Lambda^{\mathrm{loc}}, x], [\Delta P_\Lambda^{\mathrm{loc}}, y]]$. From the Wegner estimate (A.38), one has

$$\frac{1}{|\Omega|}\mathbf{E}[\mathrm{Tr}\,\chi_\Omega \Delta P_\Lambda^{\mathrm{loc}} \chi_\Omega] \le \frac{1}{|\Omega|}\mathbf{E}[\mathrm{Tr}\,\chi_\Lambda \Delta P_\Lambda^{\mathrm{loc}} \chi_\Lambda] \le \mathrm{Const} \times \Delta E. \tag{10.28}$$

Therefore it is sufficient to show that the quantity $|\Omega|^{-1}\mathbf{E}[\mathrm{Tr}\,\chi_\Omega A^* \Delta P_\Lambda^{\mathrm{loc}} A \chi_\Omega]$ is bounded. Schwarz's inequality yields

$$\begin{aligned}
&\mathrm{Tr}\,\chi_b(\mathbf{u}) A^* \Delta P_\Lambda^{\mathrm{loc}} A \chi_b(\mathbf{u}) \\
&\quad = \sum_{\mathbf{v},\mathbf{w}} \mathrm{Tr}\,\chi_b(\mathbf{u}) A^* \chi_b(\mathbf{v}) \Delta P_\Lambda^{\mathrm{loc}} \chi_b(\mathbf{w}) A \chi_b(\mathbf{u}) \\
&\quad \le \sum_{\mathbf{v},\mathbf{w}} \sqrt{\mathrm{Tr}\,\chi_b(\mathbf{u}) A^* \chi_b(\mathbf{v}) \Delta P_\Lambda^{\mathrm{loc}} \chi_b(\mathbf{v}) A \chi_b(\mathbf{u})} \times \sqrt{\mathrm{Tr}\,\chi_b(\mathbf{u}) A^* \chi_b(\mathbf{w}) \Delta P_\Lambda^{\mathrm{loc}} \chi_b(\mathbf{w}) A \chi_b(\mathbf{u})} \\
&\quad \le \mathrm{Const} \times \sum_{\mathbf{v},\mathbf{w}} \|\chi_b(\mathbf{v}) A \chi_b(\mathbf{u})\| \cdot \|\chi_b(\mathbf{w}) A \chi_b(\mathbf{u})\|,
\end{aligned} \tag{10.29}$$

where we have used the bound (8.27) for getting the second inequality. One can show that the expectation value of the right-hand side is finite in the same way as in the proof of Lemma 8.3. □

We denote by M_Λ' the event M_Λ for the Fermi energy E_{F}' in the proof of Lemma 8.1. Note that

$$\begin{aligned}
&\mathbf{E}[|\mathrm{Index}(P_{\mathrm{F}} U_{\mathbf{a}} P_{\mathrm{F}}) - (h/e^2)\sigma_{xy}^{\mathrm{in}}|\mathbf{I}(M_\Lambda \cap M_\Lambda')] \\
&\quad \le \mathbf{E}[|\mathrm{Index}(P_{\mathrm{F}} U_{\mathbf{a}} P_{\mathrm{F}}) - \mathcal{I}^\varepsilon(P_{\mathrm{F}}; \Omega, \ell_P)|] \\
&\qquad + \mathbf{E}[|\mathcal{I}^\varepsilon(P_{\mathrm{F}}; \Omega, \ell_P) - \mathcal{I}(P_{\mathrm{F}}; \Omega, \ell_P)|] + \mathbf{E}[|\mathcal{I}(P_{\mathrm{F}}; \Omega, \ell_P) - \mathcal{I}(P_{\mathrm{F},\Lambda}; \Omega)|] \\
&\qquad + \mathbf{E}[|\mathcal{I}(P_{\mathrm{F},\Lambda}; \Omega) - (h/e^2)\sigma_{xy}^{\mathrm{in}}|\mathbf{I}(M_\Lambda \cap M_\Lambda')].
\end{aligned} \tag{10.30}$$

From (8.15), (9.28) and Lemmas 9.3 and 9.7, all the terms in the right-hand side become small for large $|\Omega|, L$ and for a small ε. Further, from (8.32), the proof of Lemma 8.5 and (10.24), we have

$$\begin{aligned}&\mathbf{E}[|\sigma_{xy}^{\mathrm{in}\,\prime} - \sigma_{xy}^{\mathrm{in}}|\mathbf{I}(M_\Lambda \cap M'_\Lambda)]\\&\quad \le \mathbf{E}[|\Delta\sigma_{xy}^{\mathrm{loc}}|\mathbf{I}(M_\Lambda \cap M'_\Lambda)] + \mathbf{E}[|\sigma_{xy}^{\mathrm{out}\prime}|\mathbf{I}(M_\Lambda \cap M'_\Lambda)] + \mathbf{E}[|\sigma_{xy}^{\mathrm{out}}|\mathbf{I}(M_\Lambda \cap M'_\Lambda)]\\&\quad \le \frac{e^2}{h}\mathbf{E}|\mathcal{I}^{\mathrm{loc}}(\Delta P_\Lambda^{\mathrm{loc}})| + (\text{small correction}).\end{aligned} \tag{10.31}$$

From these observations and the fact that the indices are constant for almost every ω, we obtain

$$\begin{aligned}&|\mathrm{Index}(P'_{\mathrm{F}} U_{\mathbf{a}} P'_{\mathrm{F}}) - \mathrm{Index}(P_{\mathrm{F}} U_{\mathbf{a}} P_{\mathrm{F}})|\mathbf{E}[\mathbf{I}(M_\Lambda \cap M'_\Lambda)]\\&\quad \le \mathbf{E}[|\mathrm{Index}(P'_{\mathrm{F}} U_{\mathbf{a}} P'_{\mathrm{F}}) - \mathrm{Index}(P_{\mathrm{F}} U_{\mathbf{a}} P_{\mathrm{F}})|\mathbf{I}(M_\Lambda \cap M'_\Lambda)]\\&\quad \le \mathbf{E}|\mathcal{I}^{\mathrm{loc}}(\Delta P_\Lambda^{\mathrm{loc}})| + (\text{small correction})\\&\quad \le \mathrm{Const} \times \Delta E^{1/2} + (\text{small correction}),\end{aligned} \tag{10.32}$$

where we have used Lemma 10.2 for getting the last inequality. This implies that the index must be constant for a small change ΔE of the Fermi energy in the localization regime because the index is equal to an integer for almost every ω.

11 Widths of the Hall Conductance Plateaus

In this section, we prove that the widths of the Hall conductance plateaus are of the bulk order under certain conditions for the potentials, by estimating the number of the localized states. The conditions are realized for weak potentials as we will see in this section.

Consider first the case with $\mathbf{A}_{\mathrm{P}} = 0$. To begin with, we note that, when the strength of the random potential V_ω continuously increases from $\lambda = 0$ to $\lambda \in [-\lambda_-, \lambda_+] \subset [\lambda_{\min}, \lambda_{\max}]$, the energies E of the $n+1$-th Landau band are broadened into the interval,

$$-\|V_0^-\|_\infty - \lambda_- u_1 \le E - \mathcal{E}_n \le \|V_0^+\|_\infty + \lambda_+ u_1. \tag{11.1}$$

Let $\hat{\delta}$ be a small positive parameter. For the lower region of the band, we choose $\lambda_+ = \lambda_+^{\mathrm{low}} \le \lambda_{\max}$, $\lambda_- = \lambda_-^{\mathrm{low}}$ and $\hat{\delta}_- = \hat{\delta}$ in the condition (5.36) so that the pair $(\lambda_+, \lambda_-) = (\lambda_+^{\mathrm{low}}, \lambda_-^{\mathrm{low}})$ satisfies the condition (2.7) with a small $\lambda_- = \lambda_-^{\mathrm{low}}$. Then the condition (5.36) for the energy E leading to a localized state becomes

$$\mathcal{E}_{n-1} + \|V_0^+\|_\infty + \lambda_+^{\mathrm{low}} u_1 + \hat{\delta}\hbar\omega_c \le E \le \mathcal{E}_n - \|V_0^-\|_\infty - \lambda_-^{\mathrm{low}} u_1 - \Delta\mathcal{E}. \tag{11.2}$$

We call this interval the lower localization regime. For the upper region of the band, we choose $\lambda_+ = \lambda_+^{\mathrm{up}}$, $\lambda_- = \lambda_-^{\mathrm{up}} \ge -\lambda_{\min}$ and $\hat{\delta}_+ = \hat{\delta}/2$ in the condition (5.37) so that the pair $(\lambda_+, \lambda_-) = (\lambda_+^{\mathrm{up}}, \lambda_-^{\mathrm{up}})$ satisfies the condition (2.7) with a small $\lambda_+ = \lambda_+^{\mathrm{up}}$. The condition (5.37) for localization is

$$\mathcal{E}_n + \|V_0^+\| + \lambda_+^{\mathrm{up}} u_1 + \Delta\mathcal{E} \le E \le \mathcal{E}_{n+1} - \|V_0^-\|_\infty - \lambda_-^{\mathrm{up}} u_1 - \hat{\delta}\hbar\omega_c. \tag{11.3}$$

We call this interval the upper localization regime. We require that the positive constants, $\hat{\delta}$, λ_+^{low} and λ_-^{up}, satisfy

$$\|V_0^+\|_\infty + \|V_0^-\|_\infty + (\lambda_+^{\mathrm{low}} + \lambda_-^{\mathrm{up}}) u_1 + 2\hat{\delta}\hbar\omega_c < \hbar\omega_c \tag{11.4}$$

so that the lower and upper localization regimes overlap with each other. This condition is satisfied for a large strength B of the magnetic field for fixed strengths of the potentials. We stress that we allow the possibility that the spectral gap between two neighboring disordered-broadened Landau bands vanishes owing to the tails $\lambda \in [\lambda_{\min}, -\lambda_-^{\rm up}] \cup [\lambda_+^{\rm low}, \lambda_{\max}]$ of the coupling constants. In this situation, all the states in the $n+1$-th broadened Landau band are localized except for the energies E satisfying $-\delta E_- \le E - \mathcal{E}_n \le \delta E_+$, where $\delta E_- = \|V_0^-\|_\infty + \lambda_-^{\rm low} u_1 + \Delta\mathcal{E}$ and $\delta E_+ = \|V_0^+\|_\infty + \lambda_+^{\rm up} u_1 + \Delta\mathcal{E}$. In other words, the number of the extended states can be bounded by the number of the energy eigenvalues E satisfying $-\delta E_- \le E - \mathcal{E}_n \le \delta E_+$.

In order to obtain the upper bound for the number of the extended states, consider first the special case with $V_0 = 0$ in the Hamiltonian H_0 of (2.2). Namely, the Hamiltonian H_0 is equal to the simplest Landau Hamiltonian $H_{\rm L}$ of (3.1), combining with the present assumption $\mathbf{A}_{\rm P} = 0$. In this case, the number of the extended states can be estimated with probability nearly equal to one for the sufficiently large volume[10] by using the Wegner estimate (A.38). When $V_0 \ne 0$, the deviation of the energy eigenvalues is bounded from above by $\|V_0^+\|_\infty$ and from below by $-\|V_0^-\|_\infty$. From these observations and the min-max principle, we can estimate the number $N_{\rm ext}$ of the extended states which appear only near the center of the band as

$$N_{\rm ext} \le C_{\rm W}^{(0)} K_3^{(0)} \|g\|_\infty (\delta E_+ + \delta E_- + \|V_0^+\|_\infty + \|V_0^-\|_\infty)|\Lambda| \tag{11.5}$$

in the case with $V_0 \ne 0$, where $C_{\rm W}^{(0)}$ and $K_3^{(0)}$ are the positive constants for the case of $V_0 = 0$. Since the total number of the states in the $n+1$-th Landau band is given by $M = |\Lambda| eB/(2\pi\hbar)$, the number $N_{\rm loc}$ of the localized states in the band is evaluated as

$$N_{\rm loc} = M - N_{\rm ext} \ge B|\Lambda| \left(\frac{e}{2\pi\hbar} - C_{\rm W}^{(0)} \frac{K_3^{(0)}}{B} \|g\|_\infty \delta E \right) \tag{11.6}$$

with $\delta E = 2(\|V_0^+\|_\infty + \|V_0^-\|_\infty) + (\lambda_-^{\rm low} + \lambda_+^{\rm up}) u_1 + 2\Delta\mathcal{E}$. We note that $K_3^{(0)}/B \sim$ Const for a large B from the remark below Theorem A.2. Thus, if the strength of the potential V_0 is sufficiently weak, we can choose the parameters $\lambda_-^{\rm low}, \lambda_+^{\rm up}, \Delta\mathcal{E}$ so that the right-hand side (11.6) is strictly positive for any large magnetic field. This implies that the number $N_{\rm loc}$ is of order of the bulk. In order to discuss the case for a strong random potential which behaves like $\|u\|_\infty \sim B$ for a strong magnetic field, we recall $u_1 = 2\|u\|_\infty$. We also have $K_3^{(0)} = \mathcal{O}(1)$ which was already obtained at the end of Sect. 5. From these and the same argument, we can also get the lower bound for the number of the localized states, i.e., the width of the Hall conductance.

Let us see that the above estimate for the widths of the plateaus gives the optimal value in the limit $B \uparrow \infty$ for $V_0 = 0$. From the above bounds, we have

$$\frac{N_{\rm ext}}{M} \le \text{Const} \times [(\lambda_-^{\rm low} + \lambda_+^{\rm up}) u_1 + 2\Delta\mathcal{E}]. \tag{11.7}$$

From the argument about the initial decay estimate for the resolvent in Sect. 5, we can take the three parameters, $\lambda_-^{\rm low}, \lambda_+^{\rm up}, \Delta\mathcal{E}$, so as to go to zero in the strong magnetic field limit $B \uparrow \infty$. Thus the density of the extended states in the Landau level is vanishing in the limit.

Next consider the case with $\mathbf{A}_{\rm P} \ne 0$. The method to show the existence of the Hall conductance plateau with the width of bulk order is basically the same as in the above case with

[10] See, for example, Chap. VI of the book [31].

$\mathbf{A}_{\mathrm{P}} = 0$, except for considering strong potentials. We assume that the bump u of the random potential V_ω is written $u = \hbar\omega_c \hat{u}$ with a fixed, dimensionless function $\hat{u}$, and that the vector potential $\mathbf{A}_{\mathrm{P}}$ satisfies $\||\mathbf{A}_{\mathrm{P}}|\|_\infty \le \alpha_0 B^{1/2}$ with a small, positive constant α_0. Instead of the condition (11.4), we require that the corresponding positive constants, $\hat{\delta}$, λ_+^{low} and λ_-^{up}, satisfy

$$\begin{aligned} &\|V_0^+\|_\infty + \|V_0^-\|_\infty + (\lambda_+^{\mathrm{low}} + \lambda_-^{\mathrm{up}})u_1 + \hat{\delta}\hbar\omega_c \\ &\quad + \frac{\sqrt{2}e}{\sqrt{m_e}}\||\mathbf{A}_{\mathrm{P}}|\|_\infty(\sqrt{\mathcal{E}_n} + \sqrt{\mathcal{E}_{n+1}}) + \frac{e^2}{2m_e}\||\mathbf{A}_{\mathrm{P}}|\|_\infty^2 < \hbar\omega_c. \end{aligned} \tag{11.8}$$

Similarly, for the lower localization regime, we choose $\lambda_+ = \lambda_+^{\mathrm{low}} \le \lambda_{\max}$, $\lambda_- = \lambda_-^{\mathrm{low}}$ and $\hat{\delta}_+ = \hat{\delta}_- = \hat{\delta}/2$ in the condition (5.47) so that the pair $(\lambda_+, \lambda_-) = (\lambda_+^{\mathrm{low}}, \lambda_-^{\mathrm{low}})$ satisfies the condition (2.7) with a small $\lambda_- = \lambda_-^{\mathrm{low}}$. Then all the states in the $n+1$-th broadened Landau band with the energies $E \le \mathcal{E}_n - \delta E_-$ are localized, where

$$\delta E_- = \|V_0^-\|_\infty + \lambda_-^{\mathrm{low}} u_1 + \frac{1}{2}\hat{\delta}\hbar\omega_c + \frac{\sqrt{2}e}{\sqrt{m_e}}\||\mathbf{A}_{\mathrm{P}}|\|_\infty\sqrt{\mathcal{E}_n}. \tag{11.9}$$

For the upper localization regime, we choose $\lambda_+ = \lambda_+^{\mathrm{up}}$, $\lambda_- = \lambda_-^{\mathrm{up}} \ge -\lambda_{\min}$ and $\hat{\delta}_+ = \hat{\delta}_- = \hat{\delta}/2$ in the same condition (5.47) so that the pair $(\lambda_+, \lambda_-) = (\lambda_+^{\mathrm{up}}, \lambda_-^{\mathrm{up}})$ satisfies the condition (2.7) with a small $\lambda_+ = \lambda_+^{\mathrm{up}}$. Then all the states in the $n+1$-th broadened Landau band with the energies $E \ge \mathcal{E}_n + \delta E_+$ are localized, where

$$\delta E_+ = \|V_0^+\|_\infty + \lambda_+^{\mathrm{up}} u_1 + \frac{1}{2}\hat{\delta}\hbar\omega_c + \frac{\sqrt{2}e}{\sqrt{m_e}}\||\mathbf{A}_{\mathrm{P}}|\|_\infty\sqrt{\mathcal{E}_n} + \frac{e^2}{2m_e}\||\mathbf{A}_{\mathrm{P}}|\|_\infty^2. \tag{11.10}$$

The corresponding δE in (11.6) is given by

$$\begin{aligned} \delta E &= 2(\|V_0^+\|_\infty + \|V_0^-\|_\infty) + (\lambda_+^{\mathrm{up}} + \lambda_-^{\mathrm{low}})u_1 + \hat{\delta}\hbar\omega_c \\ &\quad + \frac{4\sqrt{2}e}{\sqrt{m_e}}\||\mathbf{A}_{\mathrm{P}}|\|_\infty\sqrt{\mathcal{E}_n} + \frac{e^2}{m_e}\||\mathbf{A}_{\mathrm{P}}|\|_\infty^2. \end{aligned} \tag{11.11}$$

Consequently there appears the Hall conductance plateau with the width of the bulk order for a fixed potential V_0, for a strong magnetic field, and for small parameters, λ_+^{up}, λ_-^{low}, $\hat{\delta}$, α_0.

12 Corrections to the Linear Response Formula

The aim of this section is to prove that both of the acceleration coefficients γ_{uy}, $u = x, y$, in the linear response formula (8.1) are vanishing in the infinite volume limit, and that the corrections $\delta\sigma_{uy}(t)$, $u = x, y$, due to the initial adiabatic process in (8.1) satisfy the bound (2.27).

We recall the expression of the acceleration coefficients [8],

$$\gamma_{uy} = \frac{e^2}{L_x L_y}\left[\frac{N}{m_e}\delta_{u,y} + \mathrm{Tr}\, v_u(P_{y,\Lambda}P_{\mathrm{F},\Lambda} + P_{\mathrm{F},\Lambda}P_{y,\Lambda})\right] \quad \text{for } u = x, y. \tag{12.1}$$

Using the partition of unity, $\{\chi_b(\mathbf{u})\}_{\mathbf{u}}$, which was introduced in Sect. 8, we have

$$\operatorname{Tr} v_u(P_{y,\Lambda}P_{\mathrm{F},\Lambda}+P_{\mathrm{F},\Lambda}P_{y,\Lambda})=\sum_{\mathbf{u}}[\operatorname{Tr} v_u P_{y,\Lambda}\chi_b(\mathbf{u})P_{\mathrm{F},\Lambda}+\operatorname{Tr} v_u P_{\mathrm{F},\Lambda}\chi_b(\mathbf{u})P_{y,\Lambda}]. \tag{12.2}$$

First let us consider the first term $\operatorname{Tr} v_u P_{y,\Lambda}\chi_b(\mathbf{u})P_{\mathrm{F},\Lambda}$ in the summand in the right-hand side. Since we can shift the location of the box $s_b(\mathbf{u})$ by using the magnetic translations, we can assume $\operatorname{dist}(\mathbf{u},\partial\Lambda)=\mathcal{O}(L)$. Write $R_\Lambda(z)=(z-H_{\omega,\Lambda})^{-1}$. Note that

$$\begin{aligned}\operatorname{Tr} v_u P_{y,\Lambda}\chi_b(\mathbf{u})P_{\mathrm{F},\Lambda}&=\frac{1}{2\pi i}\int_\gamma dz\operatorname{Tr} v_u R_\Lambda(z)v_y R_\Lambda(z)\chi_b(\mathbf{u})P_{\mathrm{F},\Lambda}\\&=\frac{1}{2\pi i}\int_\gamma dz\operatorname{Tr} v_u R_\Lambda(z)v_y\chi_\Lambda^\delta R_\Lambda(z)\chi_b(\mathbf{u})P_{\mathrm{F},\Lambda}\\&\quad+\frac{1}{2\pi i}\int_\gamma dz\operatorname{Tr} v_u R_\Lambda(z)v_y(1-\chi_\Lambda^\delta)R_\Lambda(z)\chi_b(\mathbf{u})P_{\mathrm{F},\Lambda},\end{aligned} \tag{12.3}$$

where χ_Λ^δ is the C^2, positive cutoff function which was also introduced in Sect. 8. By the same argument as in the proof of Lemma 8.2, the absolute value of the second term in the right-hand side has a stretched exponentially decaying bound as in Lemma 8.1. Since the number of $\mathbf{u}$ for the summation in the right-hand side of (12.2) is of order of the volume $L_xL_y=\mathcal{O}(L^2)$, the corresponding contribution is vanishing in the infinite volume limit $L\uparrow\infty$. Using the identity, $v_y\chi_\Lambda^\delta=(i/\hbar)[y,z-H_{\omega,\Lambda}]\chi_\Lambda^\delta$, one has

$$\begin{aligned}&\frac{1}{2\pi i}\int_\gamma dz R_\Lambda(z)v_y\chi_\Lambda^\delta R_\Lambda(z)\chi_b(\mathbf{u})\\&\quad=\frac{i}{\hbar}[P_{\mathrm{F},\Lambda},y\chi_\Lambda^\delta]\chi_b(\mathbf{u})-\frac{1}{2\pi\hbar}\int_\gamma dz R_\Lambda(z)yW(\chi_\Lambda^\delta)R_\Lambda(z)\chi_b(\mathbf{u})\end{aligned} \tag{12.4}$$

for the first term in the right-hand side of (12.3). This second term in the right-hand side also gives a small correction. In consequence, only the first term in the right-hand side of (12.4) may lead to a nonvanishing contribution in the infinite-volume limit.

Since the second term in the summand in the right-hand side of (12.2) can handled in the same way, we get

$$\begin{aligned}&\operatorname{Tr}[v_u P_{y,\Lambda}\chi_b(\mathbf{u})P_{\mathrm{F},\Lambda}+v_u P_{\mathrm{F},\Lambda}\chi_b(\mathbf{u})P_{y,\Lambda}]\\&\quad=\frac{i}{\hbar}\operatorname{Tr}\{v_u[P_{\mathrm{F},\Lambda},y\chi_\Lambda^\delta]\chi_b(\mathbf{u})P_{\mathrm{F},\Lambda}+v_u P_{\mathrm{F},\Lambda}\chi_b(\mathbf{u})[P_{\mathrm{F},\Lambda},y\chi_\Lambda^\delta]\}+\text{corrections}\\&\quad=\frac{i}{\hbar}\operatorname{Tr}\{-v_u y\chi_\Lambda^\delta P_{\mathrm{F},\Lambda}\chi_b(\mathbf{u})P_{\mathrm{F},\Lambda}+v_u P_{\mathrm{F},\Lambda}\chi_b(\mathbf{u})P_{\mathrm{F},\Lambda}y\chi_\Lambda^\delta\}+\text{corrections}\\&\quad=-\frac{1}{m_e}\delta_{u,y}\operatorname{Tr} P_{\mathrm{F},\Lambda}\chi_b(\mathbf{u})P_{\mathrm{F},\Lambda}+\text{corrections},\end{aligned} \tag{12.5}$$

where we have used $v_u y=-(i\hbar/m_e)\delta_{u,y}+yv_u$ for getting the last equality. Substituting this and (12.2) into the expression (12.1) of γ_{uy}, we obtain

$$\lim_{\Lambda^{\mathrm{sys}}\uparrow\mathbf{R}^2}\gamma_{uy}=0\quad\text{for } u=x,y \text{ with probability one.} \tag{12.6}$$

Next consider the corrections $\delta\sigma_{uy}(t)$ due to the initial adiabatic process in (8.1). We begin with recalling the expression [8],

$$\delta\sigma_{uy}(t) = \frac{ie^2}{L_x L_y}\langle \Phi_{0,\Lambda}^{(N)}, v_u^{(N)}[1 - P_{0,\Lambda}^{(N)}]\mathcal{M}_t(E_{0,\Lambda}^{(N)} - H_{\omega,\Lambda}^{(N)})v_y^{(N)}\Phi_{0,\Lambda}^{(N)}\rangle + \text{c.c.}, \tag{12.7}$$

with

$$\mathcal{M}_t(\mathcal{E}) = \left\{\left[\frac{iT}{\mathcal{E}+i\hbar\eta} - \frac{\hbar}{(\mathcal{E}+i\hbar\eta)^2}\right]e^{-\eta T}e^{i\mathcal{E}T/\hbar} + \left[\frac{\hbar}{\mathcal{E}^2} - \frac{\hbar}{(\mathcal{E}+i\hbar\eta)^2}\right]\right\}e^{i\mathcal{E}t/\hbar}, \tag{12.8}$$

where $\Phi_{0,\Lambda}^{(N)}$ is the N electron ground state vector with the energy eigenvalue $E_{0,\Lambda}^{(N)}$, $v_u^{(N)}$ the N electron velocity operator, and $P_{0,\Lambda}^{(N)}$ the projection onto the N electron ground state.

Since all the contributions can be handled in the same way, we consider

$$\begin{aligned}\mathcal{N}_{sy}(t) &:= \frac{1}{L_x L_y}\langle \Phi_{0,\Lambda}^{(N)}, v_u^{(N)}P_{\text{ex}} \\ &\quad\times [(E_{0,\Lambda}^{(N)} - H_{\omega,\Lambda}^{(N)})^{-2} - (E_{0,\Lambda}^{(N)} - H_{\omega,\Lambda}^{(N)} + i\hbar\eta)^{-2}]e^{i\hat{\theta}}v_y^{(N)}\Phi_{0,\Lambda}^{(N)}\rangle\end{aligned} \tag{12.9}$$

with $\hat{\theta} = (E_{0,\Lambda}^{(N)} - H_{\omega,\Lambda}^{(N)})t/\hbar$ as an example. Here we have written $P_{\text{ex}} = 1 - P_{0,\Lambda}^{(N)}$ for short. In order to eliminate the factor $e^{i\hat{\theta}}$, we use Schwarz's inequality. As a result, we obtain $|\mathcal{N}_{uy}(t)|^2 \le \tilde{\mathcal{N}}_u(t)\tilde{\mathcal{N}}_y(t)$ with

$$\begin{aligned}\tilde{\mathcal{N}}_u(t) &:= \frac{1}{L_x L_y}\langle \Phi_{0,\Lambda}^{(N)}, v_u^{(N)}P_{\text{ex}} \\ &\quad\times |[E_{0,\Lambda}^{(N)} - H_{\omega,\Lambda}^{(N)}]^{-1} + [E_0^{(N)} - H_{\omega,\Lambda}^{(N)} + i\hbar\eta]^{-1}|^2 v_u^{(N)}\Phi_{0,\Lambda}^{(N)}\rangle\end{aligned} \tag{12.10}$$

and

$$\begin{aligned}\tilde{\mathcal{N}}_y(t) &:= \frac{1}{L_x L_y}\langle \Phi_{0,\Lambda}^{(N)}, v_y^{(N)}P_{\text{ex}} \\ &\quad\times |[E_{0,\Lambda}^{(N)} - H_{\omega,\Lambda}^{(N)}]^{-1} - [E_{0,\Lambda}^{(N)} - H_{\omega,\Lambda}^{(N)} + i\hbar\eta]^{-1}|^2 v_y^{(N)}\Phi_{0,\Lambda}^{(N)}\rangle.\end{aligned} \tag{12.11}$$

Further the application of the inequality $\sqrt{ab} \le (a+b)/2$ for $a, b \ge 0$ yields

$$|\mathcal{N}_{uy}(t)| \le [\eta^{s/4}\tilde{\mathcal{N}}_u(t) + \eta^{-s/4}\tilde{\mathcal{N}}_y(t)]/2 \quad \text{for } s \in (0, 1/3). \tag{12.12}$$

Since the present system has no electron-electron interaction, $\tilde{\mathcal{N}}_y(t)$ is written as

$$\begin{aligned}\tilde{\mathcal{N}}_y(t) &= \frac{1}{L_x L_y}\operatorname{Tr} P_{\text{F},\Lambda}\frac{1}{2\pi i}\int_\gamma dz \frac{1}{z - H_{\omega,\Lambda}}v_y\left[\frac{1}{z - H_{\omega,\Lambda}} - \frac{1}{z - H_{\omega,\Lambda} + i\hbar\eta}\right] \\ &\quad\times \frac{1}{2\pi i}\int_\gamma dz'\left[\frac{1}{z' - H_{\omega,\Lambda}} - \frac{1}{z' - H_{\omega,\Lambda} - i\hbar\eta}\right]v_y\frac{1}{z' - H_{\omega,\Lambda}} \\ &= \frac{1}{L_x L_y}\sum_{\mathbf{a}}\operatorname{Tr} P_{\text{F},\Lambda}\frac{1}{2\pi i}\int_\gamma dz \frac{1}{z - H_{\omega,\Lambda}}v_y\left[\frac{1}{z - H_{\omega,\Lambda}} - \frac{1}{z - H_{\omega,\Lambda} + i\hbar\eta}\right] \\ &\quad\times \chi_b(\mathbf{a})\frac{1}{2\pi i}\int_\gamma dz'\left[\frac{1}{z' - H_{\omega,\Lambda}} - \frac{1}{z' - H_{\omega,\Lambda} - i\hbar\eta}\right]v_y\frac{1}{z' - H_{\omega,\Lambda}}.\end{aligned} \tag{12.13}$$

Here we have introduced the partition of unity $\{\chi_b(\mathbf{a})\}$. In the same way as the above, the summand can be further written as

$$\begin{aligned}&\operatorname{Tr} P_{\mathrm{F},\Lambda}\frac{1}{2\pi i}\int_\gamma dz[y,(z-H_{\omega,\Lambda})^{-1}]\frac{\eta}{z-H_{\omega,\Lambda}+i\hbar\eta}\\&\quad\times\chi_b(\mathbf{a})\frac{1}{2\pi i}\int_\gamma dz'\frac{\eta}{z'-H_{\omega,\Lambda}-i\hbar\eta}[(z'-H_{\omega,\Lambda})^{-1},y]+\text{correction}.\end{aligned}\tag{12.14}$$

Here the correction vanishes almost surely in the infinite volume limit by taking a suitable sequence $\{L_x, L_y\}$ of the system sizes as in the proof of Theorem 8.5. If z of the resolvent $R_\Lambda(z)=(z-H_{\omega,\Lambda})^{-1}$ is not near to the spectrum $\sigma(H_{\omega,\Lambda})$, then the resolvent is bounded and decays exponentially at large distance. Therefore we consider only the contributions from the paths near the Fermi energy E_F in the first term in the right-hand side of (12.14). As a typical example of the corresponding contributions, let us consider

$$\begin{aligned}I_\mathbf{a}:=\sum_{\mathbf{u},\mathbf{v},\mathbf{w},\mathbf{z}}&\int_{t_-}^{t_+}dt\int_{t_-}^{t_+}dt'\operatorname{Tr}\chi_b(\mathbf{u})P_{\mathrm{F},\Lambda}\chi_b(\mathbf{v})[y,R_\Lambda(E_\mathrm{F}+it)]\chi_b(\mathbf{w})\\&\times\eta R_\Lambda(E_\mathrm{F}+i(t+\hbar\eta))\chi_b(\mathbf{a})\eta R_\Lambda(E_\mathrm{F}+i(t'-\hbar\eta))\\&\times\chi_b(\mathbf{z})[R_\Lambda(E_\mathrm{F}+it'),y]\chi_b(\mathbf{u}).\end{aligned}\tag{12.15}$$

Note that

$$\begin{aligned}|\operatorname{Tr}\chi_b(\mathbf{u})P_{\mathrm{F},\Lambda}\chi_b(\mathbf{v})A|&\le\sqrt{\operatorname{Tr}\chi_b(\mathbf{u})P_{\mathrm{F},\Lambda}\chi_b(\mathbf{u})}\sqrt{\operatorname{Tr}A^*\chi_b(\mathbf{v})P_{\mathrm{F},\Lambda}\chi_b(\mathbf{v})A}\\&\le\text{Const}\times\|A\|\end{aligned}\tag{12.16}$$

for any bounded operator A, where we have used the bound (8.27), and that

$$\|\chi_b(\mathbf{v})[y,R_\Lambda(z)]\chi_b(\mathbf{w})\|\le(\text{Const}+|v_2-w_2|)\|\chi_b(\mathbf{v})R_\Lambda(z)\chi_b(\mathbf{w})\|,\tag{12.17}$$

where we have used the decomposition $y=y-y^b+y^b$ in the proof of Lemma 8.3. From these observations, we have

$$\begin{aligned}|I_\mathbf{a}|\le\text{Const}\times&\sum_{\mathbf{u},\mathbf{v},\mathbf{w},\mathbf{z}}(\text{Const}+|v_2-w_2|)(\text{Const}+|z_2-u_2|)\\&\times\int_{t_-}^{t_+}dt\|\chi_b(\mathbf{v})R_\Lambda(E_\mathrm{F}+it)\chi_b(\mathbf{w})\|\cdot\eta\|\chi_b(\mathbf{w})R_\Lambda(E_\mathrm{F}+i(t+\hbar\eta))\chi_b(\mathbf{a})\|\\&\times\int_{t_-}^{t_+}dt'\eta\|\chi_b(\mathbf{a})R_\Lambda(E_\mathrm{F}+i(t'-\hbar\eta))\\&\times\chi_b(\mathbf{z})\|\|\chi_b(\mathbf{z})R_\Lambda(E_\mathrm{F}+it')\chi_b(\mathbf{u})\|.\end{aligned}\tag{12.18}$$

The first integral is decomposed into two parts as

$$\int_{t_-}^{t_+}dt\cdots=\int_{t_-}^{-\hbar\eta/2}dt\cdots+\int_{-\hbar\eta/2}^{t_+}dt\cdots.\tag{12.19}$$

The second part of the integral is estimated as

$$\begin{aligned}
&\int_{-\hbar\eta/2}^{t_+} dt \|\chi_b(\mathbf{v})R_\Lambda(E_\mathrm{F}+it)\chi_b(\mathbf{w})\| \cdot \eta \|\chi_b(\mathbf{w})R_\Lambda(E_\mathrm{F}+i(t+\hbar\eta))\chi_b(\mathbf{a})\| \\
&\quad \le \mathrm{Const} \times \eta^{s/4} \int_{-\hbar\eta/2}^{t_+} dt \|\chi_b(\mathbf{v})R_\Lambda(E_\mathrm{F}+it)\chi_b(\mathbf{w})\|^{s/4} |t|^{s/4-1} \\
&\qquad \times \|\chi_b(\mathbf{w})R_\Lambda(E_\mathrm{F}+i(t+\hbar\eta))\chi_b(\mathbf{a})\|^{s/4},
\end{aligned} \tag{12.20}$$

where we have used the inequality $\|\chi_b(\mathbf{w})R_\Lambda(E_\mathrm{F}+i(t+\hbar\eta))\chi_b(\mathbf{a})\| \le 2/(\hbar\eta)$ for $t \ge -\hbar\eta/2$. For the first part of the integral, we can obtain a similar bound by using $\|\chi_b(\mathbf{v})R_\Lambda(E_\mathrm{F}+it)\chi_b(\mathbf{w})\| \le 2/(\hbar\eta)$ for $t \le -\hbar\eta/2$. Clearly the second integral about t' in the right-hand side of (12.18) can be treated in the same way. Combining these observations with Hölder inequality,

$$\begin{aligned}
&\mathbf{E}[\|\chi_b(\mathbf{v})R_\Lambda(E_\mathrm{F}+it)\chi_b(\mathbf{w})\|^{s/4} \|\chi_b(\mathbf{w})R_\Lambda(E_\mathrm{F}+i(t+\hbar\eta))\chi_b(\mathbf{a})\|^{s/4} \\
&\qquad \times \|\chi_b(\mathbf{a})R_\Lambda(E_\mathrm{F}+i(t'-\hbar\eta))\chi_b(\mathbf{z})\|^{s/4} \|\chi_b(\mathbf{z})R_\Lambda(E_\mathrm{F}+it')\chi_b(\mathbf{u})\|^{s/4}] \\
&\quad \le \mathbf{E}[\|\chi_b(\mathbf{v})R_\Lambda(E_\mathrm{F}+it)\chi_b(\mathbf{w})\|^{s}]^{1/4} \mathbf{E}[\|\chi_b(\mathbf{w})R_\Lambda(E_\mathrm{F}+i(t+\hbar\eta))\chi_b(\mathbf{a})\|^{s}]^{1/4} \\
&\qquad \times \mathbf{E}[\|\chi_b(\mathbf{a})R_\Lambda(E_\mathrm{F}+i(t'-\hbar\eta))\chi_b(\mathbf{z})\|^{s}]^{1/4} \\
&\qquad \times \mathbf{E}[\|\chi_b(\mathbf{z})R_\Lambda(E_\mathrm{F}+it')\chi_b(\mathbf{u})\|^{s}]^{1/4},
\end{aligned} \tag{12.21}$$

we obtain

$$\begin{aligned}
\mathbf{E}|I_\mathbf{a}| &\le \mathrm{Const} \times \eta^{s/2} \sum_{\mathbf{u},\mathbf{v},\mathbf{w},\mathbf{z}} (\mathrm{Const}+|v_2-w_2|)(\mathrm{Const}+|z_2-u_2|) \\
&\qquad \times e^{-\mu|\mathbf{v}-\mathbf{w}|/4} e^{-\mu|\mathbf{w}-\mathbf{a}|/4} e^{-\mu|\mathbf{a}-\mathbf{z}|/4} e^{-\mu|\mathbf{z}-\mathbf{u}|/4} \\
&\le \mathrm{Const} \times \eta^{s/2},
\end{aligned} \tag{12.22}$$

where we have used Fatou's lemma, Fubini–Tonelli theorem and the fractional moment bound (7.1) as in the proof of Lemma 7.1. Combining this bound, (12.13), (12.14) and (12.15), the expectation of $\tilde{\mathcal{N}}_y(t)$ of (12.12) is bounded by $\mathrm{Const} \times \eta^{s/2}$.

Since the bound,

$$|\mathcal{E}^{-1} + (\mathcal{E}+i\hbar\eta)^{-1}|^2 \le 4\mathcal{E}^{-2}, \tag{12.23}$$

holds for $\mathcal{E} \in \mathbf{R}$, the expectation of $\tilde{\mathcal{N}}_u(t)$ of (12.12) can be proved to be bounded in a easier way. As a result, we obtain that there exists a sequence $\{L_{x,n}, L_{y,n}\}_n$ of the system sizes such that the bound,

$$\lim_{L_x,L_y\to\infty} \mathcal{N}_{uy}(t) \le \mathrm{Const} \times \eta^{s/4}, \tag{12.24}$$

holds almost surely, where the positive constant may depend on ω.

Since the rest of the contributions for $\delta\sigma_{uy}(t)$ can be handled in the same way, we obtain the desired result that the bound,

$$|\delta\sigma_{uy}(t)| \le [\mathcal{C}_1(\omega) + \mathcal{C}_2(\omega)T]e^{-\eta T} + \mathcal{C}_3(\omega)\eta^{s/4}, \tag{12.25}$$

holds almost surely for $s \in (0, 1/3)$ and for $u = x, y$. Here the positive constants, $\mathcal{C}_j(\omega) < \infty$, $j = 1, 2, 3$, may depend on ω. Choosing $s = 4/13 < 1/3$, we get the bound (2.27).

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Appendix A: Wegner Estimate

In this appendix, we study the Wegner estimate [39] for the density of states for a single-electron Hamiltonian in a general setting. For this purpose, we follow the argument by Barbaroux, Combes and Hislop [18]. However, the following argument is slightly simplified compared to their original one because we are interested in two dimensions only.

Consider a single spinless electron system with a disorder potential V_ω in d dimensions. The Hamiltonian is given by $H_\omega = H_0 + V_\omega$ on $L^2(\mathbf{R}^d)$ with the unperturbed Hamiltonian,

$$H_0 = \frac{1}{2m_e}(\mathbf{p} + e\mathbf{A})^2 + V_0. \tag{A.1}$$

We assume $\mathbf{A} \in C^1(\mathbf{R}^d, \mathbf{R}^d)$ and $V_0 \in L^\infty(\mathbf{R}^d)$, and assume that the Hamiltonian H_0 is essentially self-adjoint with a boundary condition. As a disorder potential V_ω, we consider an Anderson type potential of impurities,

$$V_\omega(\mathbf{r}) = \sum_{\mathbf{a} \in \mathbf{L}^d} \lambda_{\mathbf{a}}(\omega) u(\mathbf{r} - \mathbf{a}), \tag{A.2}$$

for $\mathbf{r} = (x_1, x_2, \ldots, x_d) \in \mathbf{R}^d$. The constants $\{\lambda_{\mathbf{a}}(\omega) | \mathbf{a} \in \mathbf{L}^d\}$ form a family of independent, identically distributed random variables on a d-dimensional periodic lattice $\mathbf{L}^d \subset \mathbf{R}^d$. The common distribution has a density $g \ge 0$ which has compact support and satisfies $g \in L^\infty(\mathbf{R}) \cap C(\mathbf{R})$. We assume that the single-site potential u is non-negative and has compact support. We write the sum of the single-site potentials u as

$$\mathcal{U}(\mathbf{r}) := \sum_{\mathbf{a} \in \mathbf{L}^d} u(\mathbf{r} - \mathbf{a}) \quad \text{for } \mathbf{r} \in \mathbf{R}^d. \tag{A.3}$$

We assume $\|\mathcal{U}\|_\infty < +\infty$. Clearly this implies $\|u\|_\infty < +\infty$. From these assumptions, we have $\|V_\omega\|_\infty \le v_{\mathrm{R}} < +\infty$ with some positive constant v_{R} which is independent of the random variables.

For a bounded region $\Lambda \subset \mathbf{R}^d$, we denote by $H_{\omega,\Lambda} = H_{0,\Lambda} + V_{\omega,\Lambda}$ the Hamiltonian H_ω restricted to Λ with a boundary condition. Here $V_{\omega,\Lambda} = V_\omega | \Lambda$, i.e., $V_{\omega,\Lambda}$ is the restriction of V_ω to Λ. We assume

$$\mathcal{U}_\Lambda := \mathcal{U} | \Lambda \ge \mathcal{U}_{\min} \chi_\Lambda, \tag{A.4}$$

where $\mathcal{U}_{\min}$ is a positive constant which is independent of the bounded region Λ, and χ_Λ is the characteristic function for Λ. Namely there is no flat potential region satisfying $u = 0$. Further we assume that

$$\mathrm{Tr}(H_{0,\Lambda} + \mathcal{E}_{\min})^{-2} \chi_\Omega \le K_0 |\Omega|^{n_0} \tag{A.5}$$

for a finite region $\Omega \subset \Lambda$, with a positive constant $\mathcal{E}_{\min} > \|V_0^-\|_\infty$, where Tr stands for the trace on $L^2(\Lambda)$; K_0 and n_0 are the positive constants which are independent of the volumes $|\Lambda|, |\Omega|$ of the finite regions Λ, Ω. If the vector potential $\mathbf{A}$ satisfies the additional assumption $\mathbf{A} \in C^2(\mathbf{R}^d, \mathbf{R}^d)$, then the inequality (A.5) is valid in the dimensions $d \le 3$. See Ref. [18] for details and also for the treatment in the case of higher dimensions $d \ge 4$ in which they require a stronger assumption than the above assumption (A.5). Consider the present system with the unperturbed Hamiltonian (2.2). From the inequality (3.9) for the lower edge of the Landau band, we have

$$\mathrm{Tr}(H_{0,\Lambda}+\mathcal{E}_{\min})^{-2}\chi_{\Omega} \le \mathrm{Const} \times |b|/B \tag{A.6}$$

for strong magnetic field strengths B. Here we have assumed that the region Ω is contained in a rectangular box b such that the volume $|b|$ of the box satisfies the flux quantization condition $|b|/(2\pi\ell_B^2) \in \mathbf{N}$. Namely

$$n_0 = 1 \quad \text{and} \quad K_0 \sim \mathrm{Const} \times B^{-1} \quad \text{for a large } B. \tag{A.7}$$

Let $Q_{\omega,\Lambda}(\Delta)$ denote the spectral projection for $H_{\omega,\Lambda}$ with an energy interval $\Delta \subset \mathbf{R}$. Let ψ_E be an eigenvector of the Hamiltonian $H_{\omega,\Lambda}$, i.e., $H_{\omega,\Lambda}\psi_E = E\psi_E$ with an energy eigenvalue $E \in \mathbf{R}$. The Schrödinger equation is written as

$$(H_{0,\Lambda}+\mathcal{E}_{\min})\psi_E = (-V_{\omega,\Lambda}+E+\mathcal{E}_{\min})\psi_E. \tag{A.8}$$

Since $H_{0,\Lambda}+\mathcal{E}_{\min} > 0$ from the assumption $\mathcal{E}_{\min} > \|V_0^-\|_\infty$, one has

$$\begin{aligned}\psi_E &= \frac{1}{H_{0,\Lambda}+\mathcal{E}_{\min}}(-V_{\omega,\Lambda}+E+\mathcal{E}_{\min})\psi_E\\ &= \frac{1}{H_{0,\Lambda}+\mathcal{E}_{\min}}(-V_{\omega,\Lambda}+H_{\omega,\Lambda}+\mathcal{E}_{\min})\psi_E.\end{aligned} \tag{A.9}$$

Using this identity, one obtains

$$\begin{aligned}&\mathrm{Tr}\, Q_{\omega,\Lambda}(\Delta)\\ &\quad= \mathrm{Tr}(H_{0,\Lambda}+\mathcal{E}_{\min})^{-2}(-V_{\omega,\Lambda}+H_{\omega,\Lambda}+\mathcal{E}_{\min})Q_{\omega,\Lambda}(\Delta)(-V_{\omega,\Lambda}+H_{\omega,\Lambda}+\mathcal{E}_{\min})\\ &\quad= \mathrm{Tr}(H_{0,\Lambda}+\mathcal{E}_{\min})^{-2}V_{\omega,\Lambda}Q_{\omega,\Lambda}(\Delta)V_{\omega,\Lambda}\\ &\qquad - \mathrm{Tr}(H_{0,\Lambda}+\mathcal{E}_{\min})^{-2}V_{\omega,\Lambda}(H_{\omega,\Lambda}+\mathcal{E}_{\min})Q_{\omega,\Lambda}(\Delta)\\ &\qquad - \mathrm{Tr}(H_{0,\Lambda}+\mathcal{E}_{\min})^{-2}(H_{\omega,\Lambda}+\mathcal{E}_{\min})Q_{\omega,\Lambda}(\Delta)V_{\omega,\Lambda}\\ &\qquad + \mathrm{Tr}(H_{0,\Lambda}+\mathcal{E}_{\min})^{-2}(H_{\omega,\Lambda}+\mathcal{E}_{\min})^2 Q_{\omega,\Lambda}(\Delta).\end{aligned} \tag{A.10}$$

Let us evaluate the first term in the last line of the right-hand side. Substituting the expression (A.2) of V_ω into the term, one has

$$\begin{aligned}&\mathrm{Tr}(H_{0,\Lambda}+\mathcal{E}_{\min})^{-2}V_{\omega,\Lambda}Q_{\omega,\Lambda}(\Delta)V_{\omega,\Lambda}\\ &\quad= \sum_{\mathbf{a},\mathbf{b}} \lambda_{\mathbf{a}}\lambda_{\mathbf{b}}\,\mathrm{Tr}(H_{0,\Lambda}+\mathcal{E}_{\min})^{-2}u_{\mathbf{a},\Lambda}Q_{\omega,\Lambda}(\Delta)u_{\mathbf{b},\Lambda}\\ &\quad= \sum_{\mathbf{a},\mathbf{b}} \lambda_{\mathbf{a}}\lambda_{\mathbf{b}}\,\mathrm{Tr}\, u_{\mathbf{b},\Lambda}^{1/2}(H_{0,\Lambda}+\mathcal{E}_{\min})^{-2}u_{\mathbf{a},\Lambda}^{1/2}u_{\mathbf{a},\Lambda}^{1/2}Q_{\omega,\Lambda}(\Delta)u_{\mathbf{b},\Lambda}^{1/2},\end{aligned} \tag{A.11}$$

where we have written $u_{\mathbf{a}}(\mathbf{r}) = u(\mathbf{r}-\mathbf{a})$ and $u_{\mathbf{a},\Lambda} = u_{\mathbf{a}}|\Lambda$. Since the operator

$$\Upsilon_{\mathbf{b},\mathbf{a}}^{(1)} := u_{\mathbf{b},\Lambda}^{1/2}(H_{0,\Lambda}+\mathcal{E}_{\min})^{-2}u_{\mathbf{a},\Lambda}^{1/2} \tag{A.12}$$

is compact, there exist a pair of orthonormal bases, $\{\varphi_n^{(1)}\}_{n=1}^\infty$ and $\{\psi_n^{(1)}\}_{n=1}^\infty$, and nonnegative numbers $\{\mu_n^{(1)}\}_{n=1}^\infty$ such that[11]

$$\Upsilon_{\mathbf{b},\mathbf{a}}^{(1)} = \sum_{n=1}^{\infty} \mu_n^{(1)} \varphi_n^{(1)}(\psi_n^{(1)}, \ldots). \tag{A.13}$$

The numbers $\mu_n^{(1)}$ are the eigenvalues of $|\Upsilon_{\mathbf{b},\mathbf{a}}^{(1)}|$. Using this representation (A.13), one has

$$\begin{aligned}
&|\operatorname{Tr} \Upsilon_{\mathbf{b},\mathbf{a}}^{(1)} u_{\mathbf{a},\Lambda}^{1/2} Q_{\omega,\Lambda}(\Delta) u_{\mathbf{b},\Lambda}^{1/2}| \\
&\quad \le \sum_{n=1}^{\infty} \mu_n^{(1)} |(\psi_n^{(1)}, u_{\mathbf{a},\Lambda}^{1/2} Q_{\omega,\Lambda}(\Delta) u_{\mathbf{b},\Lambda}^{1/2} \varphi_n^{(1)})| \\
&\quad \le \frac{1}{2} \sum_{n=1}^{\infty} \mu_n^{(1)} [(\psi_n^{(1)}, u_{\mathbf{a},\Lambda}^{1/2} Q_{\omega,\Lambda}(\Delta) u_{\mathbf{a},\Lambda}^{1/2} \psi_n^{(1)}) + (\varphi_n^{(1)}, u_{\mathbf{b},\Lambda}^{1/2} Q_{\omega,\Lambda}(\Delta) u_{\mathbf{b},\Lambda}^{1/2} \varphi_n^{(1)})].
\end{aligned} \tag{A.14}$$

Therefore the expectation value of the left-hand side of the first inequality can be bounded from above as

$$\begin{aligned}
&\mathbf{E}_\Lambda[|\operatorname{Tr} \Upsilon_{\mathbf{b},\mathbf{a}}^{(1)} u_{\mathbf{a},\Lambda}^{1/2} Q_{\omega,\Lambda}(\Delta) u_{\mathbf{b},\Lambda}^{1/2}|] \\
&\quad \le \frac{1}{2} \left(\sum_{n=1}^{\infty} \mu_n^{(1)} \right) \sup_n \mathbf{E}_\Lambda [(\psi_n^{(1)}, u_{\mathbf{a},\Lambda}^{1/2} Q_{\omega,\Lambda}(\Delta) u_{\mathbf{a},\Lambda}^{1/2} \psi_n^{(1)}) \\
&\qquad + (\varphi_n^{(1)}, u_{\mathbf{b},\Lambda}^{1/2} Q_{\omega,\Lambda}(\Delta) u_{\mathbf{b},\Lambda}^{1/2} \varphi_n^{(1)})],
\end{aligned} \tag{A.15}$$

where $\mathbf{E}_\Lambda[\cdots]$ stands for the expectation with respect to the random variables on a region $\Lambda \subset \mathbf{R}^d$. The right-hand side can be evaluated by using the following Lemma A.1 which is essentially due to Kotani and Simon [47]. In order to make this paper self-contained, we give the proof of Lemma A.1 in Appendix B, following from Ref. [41].

Lemma A.1 *Let v be a nonnegative function satisfying $v \le u_{\mathbf{a},\Lambda}$. Then*

$$\left\| \int_{\mathbf{R}} d\lambda_{\mathbf{a}} g(\lambda_{\mathbf{a}}) v^{1/2} Q_{\omega,\Lambda}(\Delta) v^{1/2} \right\| \le \|g\|_\infty |\Delta|. \tag{A.16}$$

From the bound (A.16) and the inequality (A.15), one has

$$\mathbf{E}_\Lambda[|\operatorname{Tr} \Upsilon_{\mathbf{b},\mathbf{a}}^{(1)} u_{\mathbf{a},\Lambda}^{1/2} Q_{\omega,\Lambda}(\Delta) u_{\mathbf{b},\Lambda}^{1/2}|] \le \|g\|_\infty |\Delta| \|\Upsilon_{\mathbf{b},\mathbf{a}}^{(1)}\|_1, \tag{A.17}$$

where $\|\cdots\|_1 := \operatorname{Tr}|\cdots|$. Moreover, combining this bound with (A.11), one gets

$$\begin{aligned}
&\mathbf{E}_\Lambda[\operatorname{Tr}(H_{0,\Lambda} + \mathcal{E}_{\min})^{-2} V_{\omega,\Lambda} Q_{\omega,\Lambda}(\Delta) V_{\omega,\Lambda}] \\
&\quad = \sum_{\mathbf{a},\mathbf{b}} \mathbf{E}_\Lambda[\lambda_{\mathbf{a}} \lambda_{\mathbf{b}} \operatorname{Tr} \Upsilon_{\mathbf{b},\mathbf{a}}^{(1)} u_{\mathbf{a},\Lambda}^{1/2} Q_{\omega,\Lambda}(\Delta) u_{\mathbf{b},\Lambda}^{1/2}]
\end{aligned}$$

[11] See, for example, Chapter VI of the book by Reed and Simon [46].

$$\le \mathcal{M}^2 \sum_{\mathbf{a},\mathbf{b}} \mathbf{E}_\Lambda[|\operatorname{Tr} \Upsilon^{(1)}_{\mathbf{b},\mathbf{a}} u^{1/2}_{\mathbf{a},\Lambda} Q_{\omega,\Lambda}(\Delta) u^{1/2}_{\mathbf{b},\Lambda}|]$$

$$\le \mathcal{M}^2 \|g\|_\infty |\Delta| \sum_{\mathbf{a},\mathbf{b}} \|\Upsilon^{(1)}_{\mathbf{b},\mathbf{a}}\|_1, \tag{A.18}$$

where $\mathcal{M} := \sup_{\lambda\in\operatorname{supp} g} |\lambda|$.

Next consider the second term in the right-hand side of the second equality in (A.10). From the assumption (A.4), one can easily find a set of nonnegative functions $\{\tilde{u}_{\mathbf{a},\Lambda}\}_{\mathbf{a}}$ satisfying the following two conditions:

$$\tilde{u}_{\mathbf{a},\Lambda} \le u_{\mathbf{a},\Lambda} \quad \text{for any lattice site } \mathbf{a}, \tag{A.19}$$

and

$$\sum_{\mathbf{a}} \tilde{u}_{\mathbf{a},\Lambda} = \mathcal{U}_{\min} \chi_\Lambda. \tag{A.20}$$

Using the identity (A.20), one has

$$\begin{aligned}
&\operatorname{Tr}(H_{0,\Lambda} + \mathcal{E}_{\min})^{-2} V_{\omega,\Lambda}(H_{\omega,\Lambda} + \mathcal{E}_{\min}) Q_{\omega,\Lambda}(\Delta) \\
&\quad = \sum_{\mathbf{a}} \lambda_{\mathbf{a}} \operatorname{Tr}(H_{0,\Lambda} + \mathcal{E}_{\min})^{-2} u_{\mathbf{a},\Lambda}(H_{\omega,\Lambda} + \mathcal{E}_{\min}) Q_{\omega,\Lambda}(\Delta)\chi_\Lambda \\
&\quad = \frac{1}{\mathcal{U}_{\min}} \sum_{\mathbf{a},\mathbf{b}} \lambda_{\mathbf{a}} \operatorname{Tr}(H_{0,\Lambda} + \mathcal{E}_{\min})^{-2} u_{\mathbf{a},\Lambda}(H_{\omega,\Lambda} + \mathcal{E}_{\min}) Q_{\omega,\Lambda}(\Delta)\tilde{u}_{\mathbf{b},\Lambda} \\
&\quad = \frac{1}{\mathcal{U}_{\min}} \sum_{\mathbf{a},\mathbf{b}} \lambda_{\mathbf{a}} \operatorname{Tr} \Upsilon^{(2)}_{\mathbf{b},\mathbf{a}} u^{1/2}_{\mathbf{a},\Lambda}(H_{\omega,\Lambda} + \mathcal{E}_{\min}) Q_{\omega,\Lambda}(\Delta)\tilde{u}^{1/2}_{\mathbf{b},\Lambda},
\end{aligned} \tag{A.21}$$

where

$$\Upsilon^{(2)}_{\mathbf{b},\mathbf{a}} := \tilde{u}^{1/2}_{\mathbf{b},\Lambda}(H_{0,\Lambda} + \mathcal{E}_{\min})^{-2} u^{1/2}_{\mathbf{a},\Lambda}. \tag{A.22}$$

In the same way,

$$\begin{aligned}
&|\operatorname{Tr} \Upsilon^{(2)}_{\mathbf{b},\mathbf{a}} u^{1/2}_{\mathbf{a},\Lambda}(H_{\omega,\Lambda} + \mathcal{E}_{\min}) Q_{\omega,\Lambda}(\Delta)\tilde{u}^{1/2}_{\mathbf{b},\Lambda}| \\
&\quad \le \sum_{n=1}^\infty \mu^{(2)}_n |(\psi^{(2)}_n, u^{1/2}_{\mathbf{a},\Lambda}(H_{\omega,\Lambda} + \mathcal{E}_{\min}) Q_{\omega,\Lambda}(\Delta)\tilde{u}^{1/2}_{\mathbf{b},\Lambda}\varphi^{(2)}_n)| \\
&\quad \le \frac{\mathcal{E}_{\max}(\Delta)}{2} \sum_{n=1}^\infty \mu^{(2)}_n [(\psi^{(2)}_n, u^{1/2}_{\mathbf{a},\Lambda} Q_{\omega,\Lambda}(\Delta) u^{1/2}_{\mathbf{a},\Lambda}\psi^{(2)}_n) \\
&\qquad + (\varphi^{(2)}_n \tilde{u}^{1/2}_{\mathbf{b},\Lambda} Q_{\omega,\Lambda}(\Delta)\tilde{u}^{1/2}_{\mathbf{b},\Lambda}\varphi^{(2)}_n)],
\end{aligned} \tag{A.23}$$

where $\{\varphi^{(2)}_n\}_{n=1}^\infty$ and $\{\psi^{(2)}_n\}_{n=1}^\infty$ are orthonormal bases such that

$$\Upsilon^{(2)}_{\mathbf{b},\mathbf{a}} = \sum_{n=1}^\infty \mu^{(2)}_n \varphi^{(2)}_n (\psi^{(2)}_n, \ldots) \tag{A.24}$$

with the eigenvalues $\mu^{(2)}_n$ of $|\Upsilon^{(2)}_{\mathbf{b},\mathbf{a}}|$, and

$$\mathcal{E}_{\max}(\Delta) = \sup_{E\in\Delta} |E + \mathcal{E}_{\min}|. \tag{A.25}$$

Combining (A.21), (A.23) and Lemma A.1 with the condition (A.19), one has

$$\begin{aligned}
&\mathbf{E}_\Lambda[|\operatorname{Tr}(H_{0,\Lambda}+\mathcal{E}_{\min})^{-2}V_{\omega,\Lambda}(H_{\omega,\Lambda}+\mathcal{E}_{\min})Q_{\omega,\Lambda}(\Delta)|]\\
&\quad\le \mathcal{M}\frac{\mathcal{E}_{\max}(\Delta)}{\mathcal{U}_{\min}}\|g\|_\infty|\Delta|\sum_{\mathbf{a},\mathbf{b}}\|\Upsilon_{\mathbf{b},\mathbf{a}}^{(2)}\|_1.
\end{aligned}\tag{A.26}$$

In the same way,

$$\begin{aligned}
&\mathbf{E}_\Lambda[|\operatorname{Tr}(H_{0,\Lambda}+\mathcal{E}_{\min})^{-2}(H_{\omega,\Lambda}+\mathcal{E}_{\min})Q_{\omega,\Lambda}(\Delta)V_{\omega,\Lambda}|]\\
&\quad\le \mathcal{M}\frac{\mathcal{E}_{\max}(\Delta)}{\mathcal{U}_{\min}}\|g\|_\infty|\Delta|\sum_{\mathbf{a},\mathbf{b}}\|\Upsilon_{\mathbf{b},\mathbf{a}}^{(3)}\|_1
\end{aligned}\tag{A.27}$$

and

$$\begin{aligned}
&\mathbf{E}_\Lambda[|\operatorname{Tr}(H_{0,\Lambda}+\mathcal{E}_{\min})^{-2}(H_{\omega,\Lambda}+\mathcal{E}_{\min})^2Q_{\omega,\Lambda}(\Delta)|]\\
&\quad\le \left(\frac{\mathcal{E}_{\max}(\Delta)}{\mathcal{U}_{\min}}\right)^2\|g\|_\infty|\Delta|\sum_{\mathbf{a},\mathbf{b}}\|\Upsilon_{\mathbf{b},\mathbf{a}}^{(4)}\|_1,
\end{aligned}\tag{A.28}$$

where

$$\begin{aligned}
\Upsilon_{\mathbf{b},\mathbf{a}}^{(3)} &:= u_{\mathbf{b},\Lambda}^{1/2}(H_{0,\Lambda}+\mathcal{E}_{\min})^{-2}\tilde{u}_{\mathbf{a},\Lambda}^{1/2},\\
\Upsilon_{\mathbf{b},\mathbf{a}}^{(4)} &:= \tilde{u}_{\mathbf{b},\Lambda}^{1/2}(H_{0,\Lambda}+\mathcal{E}_{\min})^{-2}\tilde{u}_{\mathbf{a},\Lambda}^{1/2}.
\end{aligned}\tag{A.29}$$

Next let us estimate $\sum_{\mathbf{a},\mathbf{b}}\|\Upsilon_{\mathbf{b},\mathbf{a}}^{(i)}\|_1$, $i=1,2,3,4$. For simplicity, we consider only the case with $i=1$ because all the other cases can be estimated in the same way. We decompose the sum into two parts as

$$\sum_{\mathbf{a},\mathbf{b}}\|\Upsilon_{\mathbf{b},\mathbf{a}}^{(1)}\|_1=\sum_{\substack{\mathbf{a},\mathbf{b}\\ \text{overlap}}}\|\Upsilon_{\mathbf{b},\mathbf{a}}^{(1)}\|_1+\sum_{\substack{\mathbf{a},\mathbf{b}\\ \text{non-overlap}}}\|\Upsilon_{\mathbf{b},\mathbf{a}}^{(1)}\|_1,\tag{A.30}$$

where the first sum is over the lattice sites $\mathbf{a},\mathbf{b}$ such that the corresponding two potentials $u_{\mathbf{a},\Lambda}, u_{\mathbf{b},\Lambda}$ overlap with each other, and the second sum is over those for the non-overlapping potentials. Note that

$$\|\Upsilon_{\mathbf{b},\mathbf{a}}^{(i)}\|_1\le\sqrt{\|\Upsilon_{\mathbf{b},\mathbf{b}}^{(i)}\|_1\|\Upsilon_{\mathbf{a},\mathbf{a}}^{(i)}\|_1},\tag{A.31}$$

where we have used the inequality $\|AB\|_1\le\|A\|_2\|B\|_2$ for bounded operators A,B. Here the norm $\|\cdots\|_2$ is defined as $\|A\|_2:=\sqrt{\operatorname{Tr}A^*A}$ for a bounded operator A if the right-hand side exists. Using the inequality (A.31) and the assumption (A.5), one has

$$\sum_{\substack{\mathbf{a},\mathbf{b}\\ \text{overlap}}}\|\Upsilon_{\mathbf{b},\mathbf{a}}^{(1)}\|_1\le\text{Const}\times K_0\|u\|_\infty|\operatorname{supp}u|^{n_0}|\Lambda|,\tag{A.32}$$

where the positive constant depends only on the lattice $\mathbf{L}^d$ and on the support of the potential u, and so the constant is finite from the assumptions on the lattice and the potential.

Using Proposition C.1 in Appendix C, the second sum in the right-hand side of (A.30) can be evaluated as

$$\begin{aligned}\sum_{\substack{\mathbf{a},\mathbf{b}\\ \text{non-overlap}}} \|\Upsilon_{\mathbf{b},\mathbf{a}}^{(1)}\|_1 &\le \frac{\text{Const} \times \|u\|_\infty}{(1-\kappa^2)(\mathcal{E}_{\min} - \|V_0^-\|_\infty)} K_0 |\operatorname{supp} u|^{n_0/2} \sum_{\substack{\mathbf{a},\mathbf{b}\\ \text{non-overlap}}} r^n e^{-\alpha r} \\ &\le \frac{\text{Const} \times \|u\|_\infty}{(1-\kappa^2)(\mathcal{E}_{\min} - \|V_0^-\|_\infty)} K_0 K_2(\alpha) |\operatorname{supp} u|^{n_0/2} |\Lambda|, \end{aligned} \tag{A.33}$$

where α is given by (C.2), $r = \operatorname{dist}(\operatorname{supp} u_{\mathbf{a}}, \operatorname{supp} u_{\mathbf{b}})$, and the positive constant $K_2(\alpha)$ satisfies the bound:

$$K_2(\alpha) \le \text{Const} \times \sum_{\mathbf{b}} r^n e^{-\alpha r} \tag{A.34}$$

with the positive constant and with a fixed lattice site $\mathbf{a}$. Combining this result with the bound (A.32), one has

$$\sum_{\mathbf{a},\mathbf{b}} \|\Upsilon_{\mathbf{b},\mathbf{a}}^{(1)}\|_1 \le \text{Const} \times K_0 K_1 \|u\|_\infty |\operatorname{supp} u|^{n_0/2} |\Lambda| \tag{A.35}$$

with the positive constant

$$K_1 = |\operatorname{supp} u|^{n_0/2} + \frac{K_2(\alpha)}{(1-\kappa^2)(\mathcal{E}_{\min} - \|V_0^-\|_\infty)}. \tag{A.36}$$

Clearly this constant K_1 is independent of the strength B of the magnetic field and of the random potential V_ω. From the condition (A.19) for $\tilde{u}_{\mathbf{a},\Lambda}$, the following bounds also hold:

$$\sum_{\mathbf{a},\mathbf{b}} \|\Upsilon_{\mathbf{b},\mathbf{a}}^{(i)}\|_1 \le \text{Const} \times K_0 K_1 \|u\|_\infty |\operatorname{supp} u|^{n_0} |\Lambda| \tag{A.37}$$

for all $i = 1, 2, 3, 4$. Combining this, (A.10), (A.18), (A.26), (A.27) and (A.28), one obtains the following theorem:

Theorem A.2 *Assume the conditions (A.4) and (A.5). Let $\Delta = [E - \delta E, E + \delta E]$ be an interval of the energy with $\delta E > 0$. Then the following inequality is valid*:

$$\text{Prob}[\operatorname{dist}(\sigma(H_{\omega,\Lambda}), E) < \delta E] \le \mathbf{E}_\Lambda[\operatorname{Tr} Q_{\omega,\Lambda}(\Delta)] \le C_{\mathrm{W}} K_3 \|g\|_\infty \delta E |\Lambda| \tag{A.38}$$

with some positive constant C_{W} and with

$$K_3 = [\mathcal{M} + \mathcal{E}_{\max}(\Delta)/\mathcal{U}_{\min}]^2 K_0 K_1 \|u\|_\infty |\operatorname{supp} u|^{n_0}. \tag{A.39}$$

Here $\mathbf{E}_\Lambda[\cdots]$ is the expectation with respect to the random variables on a region $\Lambda \subset \mathbf{R}^d$.

Remark Consider the present system with the unperturbed Hamiltonian (2.2). The positive number K_0 behaves as $K_0 \sim \text{Const} \times B^{-1}$ for a large B as in (A.7). From the definition (A.25), $\mathcal{E}_{\max}(\Delta) \sim \text{Const} \times B$ for a large B. Substituting these into the above expression (A.39) of K_3, one has $K_3 \sim \text{Const} \times B$ for a large B. From this observation and the above theorem, one notices that the upper bound for the number of the states in the energy interval with a fixed width δE is proportional to the strength B of the magnetic field for a large B.

Appendix B: Proof of Lemma A.1

Following Combes and Hislop [41], we give the proof of the Kotani-Simon lemma [47].

We begin with preparing the following lemma:

Lemma B.1 *Write* $H_{\omega,\Lambda} = H'_{\omega,\Lambda} + \lambda_{\mathbf{a}} u_{\mathbf{a},\Lambda}$. *Let* v *be a nonnegative function satisfying* $u_{\mathbf{a},\Lambda} \ge v$, *and define*

$$K(\lambda, E - i\delta) = v^{1/2}(H'_{\omega,\Lambda} + \lambda u_{\mathbf{a},\Lambda} - E + i\delta)^{-1} v^{1/2} \tag{B.1}$$

for $E \in \mathbf{R}$ *and* $\delta > 0$. *Then*

$$\left\| \int_{\mathbf{R}} d\lambda \frac{\lambda_0^2}{\lambda^2 + \lambda_0^2} K(\lambda, E - i\delta) \right\| \le \pi \tag{B.2}$$

for $\lambda_0 > 0$.

Proof Since $K(\lambda, E - i\delta)$ is holomorphic in λ in the upper half-plane, one has

$$\int_{\mathbf{R}} d\lambda \frac{\lambda_0^2}{\lambda^2 + \lambda_0^2} K(\lambda, E - i\delta) = \pi \lambda_0 K(i\lambda_0, E - i\delta). \tag{B.3}$$

Note that

$$\begin{aligned} &-\operatorname{Im} K(i\lambda_0, E - i\delta) \\ &\quad = v^{1/2}(H'_{\omega,\Lambda} - i\lambda_0 u_{\mathbf{a},\Lambda} - E - i\delta)^{-1}(\lambda_0 u_{\mathbf{a},\Lambda} + \delta)(H'_{\omega,\Lambda} + i\lambda_0 u_{\mathbf{a},\Lambda} - E + i\delta)^{-1} v^{1/2} \\ &\quad \ge \lambda_0 K(i\lambda_0, E - i\delta)^* K(i\lambda_0, E - i\delta), \end{aligned} \tag{B.4}$$

where we have used the assumptions $u_{\mathbf{a},\Lambda} \ge v$ and $\delta > 0$. This implies $\|K(i\lambda_0, E - i\delta)\| \le \lambda_0^{-1}$. Combining this with the above (B.3), one has the desired result (B.2). □

Proof of Lemma A.1 Let $\tilde{\Delta} \supset \Delta$ and $\tilde{\Delta} \ne \Delta$. Using Stone's formula, one has

$$(\varphi, v^{1/2} Q_{\omega,\Lambda}(\Delta) v^{1/2}\varphi) \le \frac{1}{\pi} \lim_{\delta \downarrow 0} \operatorname{Im} \int_{\tilde{\Delta}} dE (\varphi, v^{1/2}(H_{\omega,\Lambda} - E + i\delta)^{-1} v^{1/2}\varphi) \tag{B.5}$$

for any vector φ. Further,

$$\begin{aligned} &\int_{\mathbf{R}} d\lambda_{\mathbf{a}} \frac{\lambda_0^2}{\lambda_{\mathbf{a}}^2 + \lambda_0^2} (\varphi, v^{1/2} Q_{\omega,\Lambda}(\Delta) v^{1/2}\varphi) \\ &\quad \le \frac{1}{\pi} \lim_{\delta \downarrow 0} \operatorname{Im} \int_{\tilde{\Delta}} dE \int_{\mathbf{R}} d\lambda_{\mathbf{a}} \frac{\lambda_0^2}{\lambda_{\mathbf{a}}^2 + \lambda_0^2} (\varphi, K(\lambda_{\mathbf{a}}, E - i\delta)\varphi) \le |\tilde{\Delta}| \|\varphi\|^2 \end{aligned} \tag{B.6}$$

by using Fubini's theorem and Lemma B.1. Here $\lambda_0 > 0$, and $K(\lambda, E - i\delta)$ is given by (B.1). Since $g \in L_\infty(\mathbf{R})$ with compact support from the assumption, one has

$$\left\| \int_{\mathbf{R}} d\lambda_{\mathbf{a}} g(\lambda_{\mathbf{a}}) v^{1/2} Q_{\omega,\Lambda}(\Delta) v^{1/2} \right\| \le \sup_{\lambda} g(\lambda) \frac{\lambda^2 + \lambda_0^2}{\lambda_0^2} |\Delta| \tag{B.7}$$

for any $\lambda_0 > 0$. This proves the bound (A.16). □

Appendix C: A Decay Estimate of $\Upsilon_{\mathbf{b},\mathbf{a}}^{(i)}$

In this appendix, we follow Barbaroux, Combes and Hislop [18], in order to estimate $\Upsilon_{\mathbf{b},\mathbf{a}}^{(i)}$ that appear in Appendix A. The result is summarized as the following proposition:

Proposition C.1 *Let v, w be bounded functions with a compact support, and suppose* $\mathrm{dist}(\mathrm{supp}\, v, \mathrm{supp}\, w) = r$ *with a positive distance r. Then*

$$\|v(R_{0,\Lambda})^2 w\|_1 \le \frac{\mathrm{Const} \times \|v\|_\infty \|w\|_\infty}{(1-\kappa^2)(\mathcal{E}_{\min} - \|V_0^-\|_\infty)} \times K_0(|\,\mathrm{supp}\, v|^{n_0/2} + |\,\mathrm{supp}\, w|^{n_0/2}) r^n e^{-\alpha r} \tag{C.1}$$

with

$$\alpha = \frac{\sqrt{2m_e(\mathcal{E}_{\min} - \|V_0^-\|_\infty)}}{3\hbar}\kappa \quad \textit{with } \kappa \in (0,1) \tag{C.2}$$

and with some positive number n, where $R_{0,\Lambda} = (H_{0,\Lambda} + \mathcal{E}_{\min})^{-1}$, and $\|\cdots\|_1 := \mathrm{Tr}\,|\cdots|$. The constants K_0 and n_0 are given in (A.5).

Let Ω be a region in $\mathbf{R}^d$. Then we denote by $\partial\Omega$ the boundary of Ω, and define the subset Ω_{in} of Ω as $\Omega_{\mathrm{in}} = \{\mathbf{r} \in \Omega \mid \mathrm{dist}(\mathbf{r}, \partial\Omega) > \delta\}$ with a positive δ. With a small δ, one can take three regions $\Omega^{(i)}$, $i = 1, 2, 3$ satisfying the following conditions:

$$\mathrm{supp}\, v \subset \Omega_{\mathrm{in}}^{(1)} \subset \Omega^{(1)} \subset \Omega_{\mathrm{in}}^{(2)} \subset \Omega^{(2)} \subset \Omega_{\mathrm{in}}^{(3)} \subset \Omega^{(3)} \subset \Lambda, \tag{C.3}$$

$$|\Omega^{(2)}| \le \mathrm{Const} \times r^d, \tag{C.4}$$

$$\mathrm{dist}(\mathrm{supp}\, w, \Omega^{(3)}) > r/3, \tag{C.5}$$

and

$$\mathrm{dist}(\mathrm{supp}\, v, \Gamma^{(1)}) > r/3, \tag{C.6}$$

where $\Gamma^{(1)} = \Omega^{(1)} \backslash \Omega_{\mathrm{in}}^{(1)}$, and we also write $\Gamma^{(i)} = \Omega^{(i)} \backslash \Omega_{\mathrm{in}}^{(i)}$ for $i = 2, 3$.

Let $\tilde{\chi}_i \in C^2(\Lambda)$, $i = 1, 2, 3$, be three nonnegative functions satisfying

$$\tilde{\chi}_i|_{\Omega_{\mathrm{in}}^{(i)}} = 1 \quad \text{and} \quad \tilde{\chi}_i|_{\Lambda\backslash\Omega^{(i)}} = 0. \tag{C.7}$$

In the following, we denote by χ_i' the characteristic function $\chi_{\Gamma^{(i)}}$ of the region $\Gamma^{(i)}$ for $i = 1, 2, 3$, and write $R_{0,i} = R_{0,\Omega^{(i)}} = (H_{0,\Omega^{(i)}} + \mathcal{E}_{\min})^{-1}$ for $i = 1, 2, 3$. Next introduce the geometric resolvent equation,

$$\tilde{\chi}_i R_{0,\Lambda} = R_{0,i}\tilde{\chi}_i + R_{0,i} W(\tilde{\chi}_i) R_{0,\Lambda} \tag{C.8}$$

for $i = 1, 2, 3$, where $W(\cdots)$ is given by (5.7).

Using (C.8) and $v\tilde{\chi}_2 = v$, one has

$$v(R_{0,\Lambda})^2 w = v\tilde{\chi}_2(R_{0,\Lambda})^2 w = vR_{0,2}\tilde{\chi}_2 R_{0,\Lambda} w + vR_{0,2} W(\tilde{\chi}_2)(R_{0,\Lambda})^2 w. \tag{C.9}$$

The first term in the right-hand side can be rewritten as

$$\begin{aligned} vR_{0,2}\tilde{\chi}_2 R_{0,\Lambda} w &= v(R_{0,2})^2 W(\tilde{\chi}_2) R_{0,\Lambda} w \\ &= v(R_{0,2})^2 W(\tilde{\chi}_2)\tilde{\chi}_3 R_{0,\Lambda} w \\ &= v(R_{0,2})^2 W(\tilde{\chi}_2) R_{0,3} W(\tilde{\chi}_3) R_{0,\Lambda} w \end{aligned} \tag{C.10}$$

by using $\tilde{\chi}_2 w = \tilde{\chi}_3 w = 0$, $W(\tilde{\chi}_2)\tilde{\chi}_3 = W(\tilde{\chi}_2)$ and the geometric resolvent equation (C.8). Therefore

$$\begin{aligned} \|vR_{0,2}\tilde{\chi}_2 R_{0,\Lambda} w\|_1 &= \|v(R_{0,2})^2 W(\tilde{\chi}_2) R_{0,3} W(\tilde{\chi}_3) R_{0,\Lambda} w\|_1 \\ &\le \|vR_{0,2}\|_2 \|R_{0,2} W(\tilde{\chi}_2) R_{0,3} W(\tilde{\chi}_3) R_{0,\Lambda} w\|_2 \\ &\le \|vR_{0,2}\|_2 \|wR_{0,\Lambda} W^*(\tilde{\chi}_3) R_{0,3} W^*(\tilde{\chi}_2) R_{0,2}\|_2 \\ &= \|vR_{0,2}\|_2 \|wR_{0,\Lambda}\chi_3' W^*(\tilde{\chi}_3) R_{0,3} W^*(\tilde{\chi}_2) R_{0,2}\|_2 \\ &\le \|vR_{0,2}\|_2 \|wR_{0,\Lambda}\chi_3'\| \|W^*(\tilde{\chi}_3) R_{0,3} W^*(\tilde{\chi}_2) R_{0,2}\|_2, \end{aligned} \tag{C.11}$$

where we have used $\chi_3' W^*(\tilde{\chi}_3) = W^*(\tilde{\chi}_3)$, the equality $\|A^*\|_2 = \|A\|_2$ and the inequality $\|AB\|_1 \le \|A\|_2 \|B\|_2$ for bounded operators A, B. The norm $\|\cdots\|_2$ is defined as $\|A\|_2 := \sqrt{\operatorname{Tr} A^*A}$ for a bounded operator A.

The second term in the right-hand side of (C.9) can be written as

$$\begin{aligned} vR_{0,2} W(\tilde{\chi}_2)(R_{0,\Lambda})^2 w &= v\tilde{\chi}_1 R_{0,2} W(\tilde{\chi}_2)(R_{0,\Lambda})^2 w \\ &= vR_{0,1} W(\tilde{\chi}_1) R_{0,2} W(\tilde{\chi}_2)(R_{0,\Lambda})^2 w \end{aligned} \tag{C.12}$$

by using $v\tilde{\chi}_1 = v$, $\tilde{\chi}_1 W(\tilde{\chi}_2) = 0$, and the geometric resolvent equation

$$\tilde{\chi}_1 R_{0,2} = R_{0,1}\tilde{\chi}_1 + R_{0,1} W(\tilde{\chi}_1) R_{0,2}. \tag{C.13}$$

Therefore the norm can be evaluated as

$$\begin{aligned} \|vR_{0,2} W(\tilde{\chi}_2)(R_{0,\Lambda})^2 w\|_1 &= \|vR_{0,1} W(\tilde{\chi}_1) R_{0,2} W(\tilde{\chi}_2)(R_{0,\Lambda})^2 w\|_1 \\ &= \|vR_{0,1}\chi_1' W(\tilde{\chi}_1) R_{0,2} W(\tilde{\chi}_2)(R_{0,\Lambda})^2 w\|_1 \\ &\le \|vR_{0,1}\chi_1'\| \|W(\tilde{\chi}_1) R_{0,2} W(\tilde{\chi}_2)(R_{0,\Lambda})^2 w\|_1 \\ &\le \|vR_{0,1}\chi_1'\| \|W(\tilde{\chi}_1) R_{0,2} W(\tilde{\chi}_2) R_{0,\Lambda}\|_2 \|R_{0,\Lambda} w\|_2, \end{aligned} \tag{C.14}$$

where we have used the identity $\chi_1' W(\tilde{\chi}_1) = W(\tilde{\chi}_1)$.

Combining (C.9), (C.11) and (C.14), one has

$$\begin{aligned} \|v(R_{0,\Lambda})^2 w\|_1 &\le \|vR_{0,2}\tilde{\chi}_2 R_{0,\Lambda} w\|_1 + \|vR_{0,2} W(\tilde{\chi}_2)(R_{0,\Lambda})^2 w\|_1 \\ &\le \|vR_{0,2}\|_2 \|wR_{0,\Lambda}\chi_3'\| \|W^*(\tilde{\chi}_3) R_{0,3} W^*(\tilde{\chi}_2) R_{0,2}\|_2 \\ &\quad + \|R_{0,\Lambda} w\|_2 \|vR_{0,1}\chi_1'\| \|W(\tilde{\chi}_1) R_{0,2} W(\tilde{\chi}_2) R_{0,\Lambda}\|_2. \end{aligned} \tag{C.15}$$

In order to estimate the right-hand side, we use the following lemma:

Lemma C.2 *Let φ be a vector in the domain of the Hamiltonian $H_{0,\Omega^{(2)}}$. Then*

$$\|W^*(\tilde{\chi}_3)R_{0,3}W^*(\tilde{\chi}_2)\varphi\| \le \text{Const} \times \|\varphi\|, \tag{C.16}$$

where the positive constant in the right-hand side depends only on the cut-off functions, $\tilde{\chi}_2$ and $\tilde{\chi}_3$.

Proof Note that

$$\|W^*(\tilde{\chi}_3)R_{0,3}W^*(\tilde{\chi}_2)\varphi\| \le \|W^*(\tilde{\chi}_3)R_{0,3}^{1/2}\|\|R_{0,3}^{1/2}W^*(\tilde{\chi}_2)\varphi\|. \tag{C.17}$$

Using (5.9), one has

$$\|W^*(\tilde{\chi}_3)R_{0,3}^{1/2}\| \le \frac{\hbar}{m_e}\|(\nabla\tilde{\chi}_3)\cdot(\mathbf{p}+e\mathbf{A})R_{0,3}^{1/2}\| + \frac{\hbar^2}{2m_e}\|(\Delta\tilde{\chi}_3)R_{0,3}^{1/2}\|. \tag{C.18}$$

The first term in the right-hand side can be estimated as follows: Using the Schwarz inequality, one has

$$\begin{aligned}
&(\psi, R_{0,3}^{1/2}(\mathbf{p}+e\mathbf{A})\cdot(\nabla\tilde{\chi}_3)(\nabla\tilde{\chi}_3)\cdot(\mathbf{p}+e\mathbf{A})R_{0,3}^{1/2}\psi)\\
&\quad\le \sqrt{(\psi, R_{0,3}^{1/2}(\mathbf{p}+e\mathbf{A})^2R_{0,3}^{1/2}\psi)}\\
&\qquad\times\sqrt{(\psi, R_{0,3}^{1/2}(\mathbf{p}+e\mathbf{A})\cdot(\nabla\tilde{\chi}_3)|\nabla\tilde{\chi}_3|^2(\nabla\tilde{\chi}_3)\cdot(\mathbf{p}+e\mathbf{A})R_{0,3}^{1/2}\psi)}\\
&\quad\le \sqrt{2m_e}\||\nabla\tilde{\chi}_3|\|_\infty\|(\nabla\tilde{\chi}_3)\cdot(\mathbf{p}+e\mathbf{A})R_{0,3}^{1/2}\psi\|\|\psi\|
\end{aligned}\tag{C.19}$$

for any vector ψ, where we have used

$$R_{0,3}^{1/2}\frac{1}{2m_e}(\mathbf{p}+e\mathbf{A})^2R_{0,3}^{1/2} \le 1 \tag{C.20}$$

which is derived from the assumption $\mathcal{E}_{\min} > \|V_0^-\|_\infty$. As a result, one obtain

$$\|(\nabla\tilde{\chi}_3)\cdot(\mathbf{p}+e\mathbf{A})R_{0,3}^{1/2}\| \le \sqrt{2m_e}\||\nabla\tilde{\chi}_3|\|_\infty. \tag{C.21}$$

Substituting this into the right-hand side of (C.18), one gets

$$\|W^*(\tilde{\chi}_3)R_{0,3}^{1/2}\| \le \hbar\sqrt{\frac{m_e}{2}}\||\nabla\tilde{\chi}_3|\|_\infty + \frac{\hbar^2}{2m_e}\frac{\|\Delta\tilde{\chi}_3\|_\infty}{\sqrt{\mathcal{E}_{\min}-\|V_0^-\|_\infty}}. \tag{C.22}$$

Using (5.9) again, one has

$$\|R_{0,3}^{1/2}W^*(\tilde{\chi}_2)\varphi\| \le \frac{\hbar}{m_e}\|R_{0,3}^{1/2}(\mathbf{p}+e\mathbf{A})\cdot(\nabla\tilde{\chi}_2)\varphi\| + \frac{\hbar^2}{2m_e}\|R_{0,3}^{1/2}(\Delta\tilde{\chi}_2)\varphi\|. \tag{C.23}$$

The norm of the first term in the right-hand side can be evaluated as

$$\begin{aligned}
&(\varphi, (\nabla\tilde{\chi}_2)\cdot(\mathbf{p}+e\mathbf{A})R_{0,3}(\mathbf{p}+e\mathbf{A})\cdot(\nabla\tilde{\chi}_2)\varphi)\\
&\quad\le \||\nabla\tilde{\chi}_2|\|_\infty\|\varphi\|\sqrt{(\varphi, (\nabla\tilde{\chi}_2)\cdot(\mathbf{p}+e\mathbf{A})R_{0,3}(\mathbf{p}+e\mathbf{A})^2R_{0,3}(\mathbf{p}+e\mathbf{A})\cdot(\nabla\tilde{\chi}_2)\varphi)}\\
&\quad\le \sqrt{2m_e}\||\nabla\tilde{\chi}_2|\|_\infty\|\varphi\|\|R_{0,3}^{1/2}(\mathbf{p}+e\mathbf{A})\cdot(\nabla\tilde{\chi}_2)\varphi\|,
\end{aligned}\tag{C.24}$$

where we have used the Schwarz inequality and the inequality (C.20). Therefore,

$$\|R_{0,3}^{1/2}(\mathbf{p}+e\mathbf{A})\cdot(\nabla\tilde{\chi}_2)\varphi\| \le \sqrt{2m_e}\||\nabla\tilde{\chi}_2|\|_\infty\|\varphi\|. \tag{C.25}$$

Substituting this into the right-hand side of (C.23), one gets

$$\|R_{0,3}^{1/2}W^*(\tilde{\chi}_2)\varphi\| \le \left\{\hbar\sqrt{\frac{m_e}{2}}\||\nabla\tilde{\chi}_2|\|_\infty + \frac{\hbar^2}{2m_e}\frac{\|\Delta\tilde{\chi}_2\|_\infty}{\sqrt{\mathcal{E}_{\min}-\|V_0^-\|_\infty}}\right\}\|\varphi\|. \tag{C.26}$$

The bound (C.16) follows from (C.17), (C.22) and (C.26). □

From this lemma, immediately one gets

$$\|W^*(\tilde{\chi}_3)R_{0,3}W^*(\tilde{\chi}_2)R_{0,2}\|_2 \le \text{Const} \times \|\chi_2' R_{0,2}\|_2. \tag{C.27}$$

In the same way,

$$\|W(\tilde{\chi}_1)R_{0,2}W(\tilde{\chi}_2)R_{0,\Lambda}\|_2 \le \text{Const} \times \|\chi_2' R_{0,\Lambda}\|_2. \tag{C.28}$$

Substituting these two bounds into (C.15), one gets

$$\begin{aligned}\|v(R_{0,\Lambda})^2 w\|_1 \le{}& \text{Const} \times [\|vR_{0,2}\|_2\|\chi_2' R_{0,2}\|_2\|wR_{0,\Lambda}\chi_3'\| \\ &+ \|R_{0,\Lambda}w\|_2\|\chi_2' R_{0,\Lambda}\|_2\|vR_{0,1}\chi_1'\|].\end{aligned} \tag{C.29}$$

Note that the ground state energy E_0 of the Hamiltonian $H_{0,\Lambda}$ satisfies $E_0 \ge -\|V_0^-\|_\infty$. Taking

$$\beta = \frac{\sqrt{2m_e(\mathcal{E}_{\min}-\|V_0^-\|_\infty)}}{\hbar}\kappa \quad \text{with } \kappa \in (0,1) \tag{C.30}$$

in the bound (D.32) in the next Appendix D, one has

$$\|vR_{0,\Omega}w\| \le \frac{\|v\|_\infty\|w\|_\infty}{(1-\kappa^2)(\mathcal{E}_{\min}-\|V_0^-\|_\infty)}e^{-\beta r} \quad \text{for a region } \Omega, \tag{C.31}$$

where r is the distance between the supports of the two functions v and w. Combining this inequality, (C.5), (C.6) and (C.29), one has

$$\begin{aligned}\|v(R_{0,\Lambda})^2 w\|_1 \le{}& \frac{\text{Const} \times \|v\|_\infty\|w\|_\infty}{(1-\kappa^2)(\mathcal{E}_{\min}-\|V_0^-\|_\infty)} \\ &\times \left[\frac{\|vR_{0,2}\|_2}{\|v\|_\infty}\|\chi_2' R_{0,2}\|_2 + \frac{\|R_{0,\Lambda}w\|_2}{\|w\|_\infty}\|\chi_2' R_{0,\Lambda}\|_2\right]e^{-\alpha r},\end{aligned} \tag{C.32}$$

where $\alpha = \beta/3$ with the above β of (C.30). Further,

$$\begin{aligned}\|v(R_{0,\Lambda})^2 w\|_1 \le{}& \frac{\text{Const} \times \|v\|_\infty\|w\|_\infty}{(1-\kappa^2)(\mathcal{E}_{\min}-\|V_0^-\|_\infty)} \\ &\times K_0(|\operatorname{supp} v|^{n_0/2} + |\operatorname{supp} w|^{n_0/2})|\Omega^{(2)}\backslash\Omega_{\text{in}}^{(2)}|^{n_0/2}e^{-\alpha r}\end{aligned} \tag{C.33}$$

by using the assumption (A.5). Thus one gets the desired bound (C.1) from the bound (C.4) on the region $\Omega^{(2)}$.

Appendix D: Decay Estimates of Resolvents

In this appendix, the exponential decay bound for the resolvent $(H_\omega - z)^{-1}$ is obtained by using the Combes-Thomas method [40].

D.1 The General Case

For the general case with $\mathbf{A}_\mathrm{P} \neq 0$, we use the improved version [18] of the Combes-Thomas method [40]. The results are given by Theorem D.2 and the bound (D.32) below.

Consider the d-dimensional Hamiltonian,

$$H_\Lambda = \frac{1}{2m_e}(\mathbf{p} + e\mathbf{A})^2 + V_\Lambda, \tag{D.1}$$

with a general vector potential $\mathbf{A} \in C^1(\Lambda, \mathbf{R}^d)$ and a general electrostatic potential $V_\Lambda \in L^\infty(\Lambda)$. Let $j_\delta \in C_0^\infty(\mathbf{R}^d)$ satisfying $j_\delta \geq 0$, $\operatorname{supp} j_\delta \subset \{\mathbf{r} \mid |\mathbf{r}| \leq \delta\}$ with a small δ and $\int_{\mathbf{R}^d} j_\delta(\mathbf{r}) dx_1 \cdots dx_d = 1$. Let Ω be a bounded region with smooth boundary, and define $\tilde{\rho}(\mathbf{r}) = \operatorname{dist}(\mathbf{r}, \Omega)$. Following [15], we introduce the smooth distance function $\rho(\mathbf{r}) = j_\delta * \tilde{\rho}(\mathbf{r})$. Note that, for $\beta > 0$,

$$\begin{aligned} e^{-\beta\rho} H_\Lambda e^{\beta\rho} &= \frac{1}{2m_e}(\mathbf{p} + e\mathbf{A} - i\hbar\beta\nabla\rho)^2 + V_\Lambda \\ &= H_\Lambda - \frac{\hbar^2\beta^2}{2m_e}(\nabla\rho)^2 - \frac{i\hbar\beta}{2m_e}[\nabla\rho \cdot (\mathbf{p} + e\mathbf{A}) + (\mathbf{p} + e\mathbf{A}) \cdot \nabla\rho]. \end{aligned} \tag{D.2}$$

We write

$$e^{-\beta\rho} H_\Lambda e^{\beta\rho} = \tilde{H}_\Lambda + i\beta J \tag{D.3}$$

with

$$\begin{aligned} \tilde{H}_\Lambda &= H_\Lambda - \frac{\hbar^2\beta^2}{2m_e}(\nabla\rho)^2, \\ J &= -\frac{\hbar}{2m_e}[\nabla\rho \cdot (\mathbf{p} + e\mathbf{A}) + (\mathbf{p} + e\mathbf{A}) \cdot \nabla\rho]. \end{aligned} \tag{D.4}$$

We take $\tilde{C}_0 > 0$ satisfying

$$-\|V_\Lambda^-\|_\infty - \frac{\hbar^2\beta^2}{2m_e}\||\nabla\rho|\|_\infty^2 + \tilde{C}_0 > 0, \tag{D.5}$$

where $V_\Lambda^\pm = \max\{\pm V_\Lambda, 0\}$. Then one has

$$\tilde{H}_\Lambda + \tilde{C}_0 \geq C_0 > 0 \tag{D.6}$$

with some constant C_0. We define

$$X_{E+i\varepsilon} = \frac{\tilde{H}_\Lambda - E - i\varepsilon}{\tilde{H}_\Lambda + \tilde{C}_0} \quad \text{for } E, \varepsilon \in \mathbf{R}, \tag{D.7}$$

and define

$$Y = (\tilde{H}_\Lambda + \tilde{C}_0)^{-1/2} J (\tilde{H}_\Lambda + \tilde{C}_0)^{-1/2}. \tag{D.8}$$

Let us estimate the norm $\|Y\|$ of this operator. From the expression (D.4) of the operator J, one has

$$\|Y\| \le \frac{\hbar}{2m_e}[\|\tilde{R}^{1/2}\nabla\rho\cdot(\mathbf{p}+e\mathbf{A})\tilde{R}^{1/2}\| + \|\tilde{R}^{1/2}(\mathbf{p}+e\mathbf{A})\cdot\nabla\rho\tilde{R}^{1/2}\|], \tag{D.9}$$

where we have written $\tilde{R}=(\tilde{H}_\Lambda+\tilde{C}_0)^{-1}$. Take $\psi=\tilde{R}^{1/2}\varphi$ with $\varphi\in L^2(\Lambda)$. Using the Schwarz inequality and (D.6), one has

$$\begin{aligned}
&(\psi,(\mathbf{p}+e\mathbf{A})\cdot\nabla\rho\tilde{R}\nabla\rho\cdot(\mathbf{p}+e\mathbf{A})\psi)\\
&\quad\le\sqrt{(\psi,(\mathbf{p}+e\mathbf{A})^2\psi)}\times\sqrt{(\psi,(\mathbf{p}+e\mathbf{A})\cdot\nabla\rho\tilde{R}|\nabla\rho|^2\tilde{R}\nabla\rho\cdot(\mathbf{p}+e\mathbf{A})\psi)}\\
&\quad\le\sqrt{2m_e}\||\nabla\rho|\|_\infty\|\varphi\|\sqrt{(\psi,(\mathbf{p}+e\mathbf{A})\cdot\nabla\rho\tilde{R}^2\nabla\rho\cdot(\mathbf{p}+e\mathbf{A})\psi)}\\
&\quad\le\sqrt{\frac{2m_e}{C_0}}\||\nabla\rho|\|_\infty\|\varphi\|\sqrt{(\psi,(\mathbf{p}+e\mathbf{A})\cdot\nabla\rho\tilde{R}\nabla\rho\cdot(\mathbf{p}+e\mathbf{A})\psi)}.
\end{aligned}\tag{D.10}$$

Therefore

$$(\psi,(\mathbf{p}+e\mathbf{A})\cdot\nabla\rho\tilde{R}\nabla\rho\cdot(\mathbf{p}+e\mathbf{A})\psi)\le\frac{2m_e}{C_0}\||\nabla\rho|\|_\infty^2\|\varphi\|^2. \tag{D.11}$$

Similarly

$$\begin{aligned}
&(\psi,\nabla\rho\cdot(\mathbf{p}+e\mathbf{A})\tilde{R}(\mathbf{p}+e\mathbf{A})\cdot\nabla\rho\psi)\\
&\quad\le\||\nabla\rho|\|_\infty\|\psi\|\sqrt{(\psi,\nabla\rho\cdot(\mathbf{p}+e\mathbf{A})\tilde{R}(\mathbf{p}+e\mathbf{A})^2\tilde{R}(\mathbf{p}+e\mathbf{A})\cdot\nabla\rho\psi)}\\
&\quad\le\sqrt{2m_e}\||\nabla\rho|\|_\infty\|\psi\|\sqrt{(\psi,\nabla\rho\cdot(\mathbf{p}+e\mathbf{A})\tilde{R}(\mathbf{p}+e\mathbf{A})\cdot\nabla\rho\psi)}.
\end{aligned}\tag{D.12}$$

This implies

$$\begin{aligned}
&(\psi,\nabla\rho\cdot(\mathbf{p}+e\mathbf{A})\tilde{R}(\mathbf{p}+e\mathbf{A})\cdot\nabla\rho\psi)\\
&\quad\le 2m_e\||\nabla\rho|\|_\infty^2\|\psi\|^2\le\frac{2m_e}{C_0}\||\nabla\rho|\|_\infty^2\|\varphi\|^2.
\end{aligned}\tag{D.13}$$

Substituting these bounds into (D.9), one has

$$\|Y\|\le\frac{\sqrt{2}\hbar}{\sqrt{m_eC_0}}\||\nabla\rho|\|_\infty. \tag{D.14}$$

Consider the situation that the Hamiltonian H_Λ has a spectral gap (E_-,E_+), and we take $E\in(E_-,E_+)$. We define $d_\pm:=\mathrm{dist}(\sigma(X_E)\cap\mathbf{R}^\pm,0)$, and $u_\pm=P_\pm u$, where $P_\pm$ is the spectral projections for X_E onto the subspaces corresponding to the sets $\sigma(X_E)\cap\mathbf{R}^\pm$, respectively. We take β satisfying $E_+-E>\hbar^2\beta^2\||\nabla\rho|\|_\infty^2/(2m_e)$. Then one has

$$d_+>\frac{E_+-E-\hbar^2\beta^2\||\nabla\rho|\|_\infty^2/(2m_e)}{E_++\tilde{C}_0}=:\delta_+ \tag{D.15}$$

and

$$d_->\frac{E-E_-}{E_-+\tilde{C}_0}=:\delta_-. \tag{D.16}$$

Lemma D.1 *Suppose that the parameter* β *satisfies the condition,*

$$0 < \beta < \frac{\sqrt{2m_e C_0}}{\hbar \||\nabla\rho|\|_\infty} \min\left\{\frac{1}{4}\sqrt{\delta_+\delta_-}, \sqrt{\frac{E_+ - E}{C_0}}\right\}. \tag{D.17}$$

Then

$$\|(X_{E+i\varepsilon} + i\beta Y)u\| \geq \frac{1}{2}\min\{d_+, d_-\}\|u\| \tag{D.18}$$

for $\varepsilon \in \mathbf{R}$.

Proof From the bound (D.14) for $\|Y\|$ and the assumption on β, one has

$$\beta\|Y\| \leq \frac{1}{2}\sqrt{\delta_+\delta_-} \leq \frac{1}{2}\sqrt{d_+d_-}. \tag{D.19}$$

Note that $X_{E+i\varepsilon} = X_E - i\varepsilon\tilde{R}$ with $\tilde{R} = (\tilde{H}_\Lambda + \tilde{C}_0)^{-1}$. Using this and the Schwarz inequality, one gets

$$\begin{aligned}\|u\|\|(X_{E+i\varepsilon} + i\beta Y)u\| &\geq \mathrm{Re}((u_+ - u_-), (X_E - i\varepsilon\tilde{R} + i\beta Y)(u_+ + u_-)) \\ &\geq d_+\|u_+\|^2 + d_-\|u_-\|^2 - 2\beta\,\mathrm{Im}(u_+, Yu_-) \\ &\geq \frac{1}{2}(d_+\|u_+\|^2 + d_-\|u_-\|^2).\end{aligned} \tag{D.20}$$

This implies the desired bound. □

We write

$$\beta = \frac{\sqrt{2m_e}}{\hbar\||\nabla\rho|\|_\infty}\sqrt{E_+ - E} \times \kappa \tag{D.21}$$

in terms of the parameter $\kappa \in (0, 1)$. Substituting this into (D.15), one has

$$\delta_+ = \frac{E_+ - E}{E_+ + \tilde{C}_0}(1 - \kappa^2). \tag{D.22}$$

Further, by substituting these into the bound (D.17), the maximum value of κ satisfying the bound is obtained as

$$\kappa = \sqrt{\frac{C_0(E - E_-)}{C_0(E - E_-) + 16(E_+ + \tilde{C}_0)(E_- + \tilde{C}_0)}} < 1. \tag{D.23}$$

As a result, we can take

$$\beta = \frac{\sqrt{2m_e}}{\hbar\||\nabla\rho|\|_\infty}\sqrt{\frac{C_0(E_+ - E)(E - E_-)}{C_0(E - E_-) + 16(E_+ + \tilde{C}_0)(E_- + \tilde{C}_0)}}. \tag{D.24}$$

Theorem D.2 *Let* E *be in the spectral gap* $(E_-, E_+) \subset \mathbf{R}$ *of the Hamiltonian* H_Λ. *Let* v, w *be bounded functions with a compact support. Suppose that the boundary of* supp v *is smooth. Then*

$$\|v(H_\Lambda - E - i\varepsilon)^{-1}w\| \leq C_1\|v\|_\infty\|w\|_\infty e^{-\beta r} \quad \textit{for } \varepsilon \in \mathbf{R}, \tag{D.25}$$

where $r = \mathrm{dist}(\mathrm{supp}\, v, \mathrm{supp}\, w)$, β *is given by* (D.24), *and*

$$C_1 = \mathrm{Const} \times C_0^{-1} \max\left\{ \frac{E_+ + \tilde{C}_0}{(1-\kappa^2)(E_+ - E)}, \frac{E_- + \tilde{C}_0}{E - E_-} \right\} \tag{D.26}$$

with κ *given by* (D.23). *The two real constants* C_0 *and* $\tilde{C}_0$ *satisfy the conditions* $\tilde{H} + \tilde{C}_0 > C_0 > 0$ *and* (D.5).

Proof Let φ be in the domain of the operator $\tilde{H}_\Lambda$. Then

$$\begin{aligned} \|(\tilde{H}_\Lambda + i\beta J - E - i\varepsilon)\varphi\| &= \|(\tilde{H}_\Lambda + \tilde{C}_0)^{1/2}(X_{E+i\varepsilon} + i\beta Y)(\tilde{H}_\Lambda + \tilde{C}_0)^{1/2}\varphi\| \\ &\ge C_0^{1/2}\|(X_{E+i\varepsilon} + i\beta Y)(\tilde{H}_\Lambda + \tilde{C}_0)^{1/2}\varphi\| \\ &\ge \frac{1}{2}C_0^{1/2}\min\{d_+, d_-\}\|(\tilde{H}_\Lambda + \tilde{C}_0)^{1/2}\varphi\| \\ &\ge \frac{1}{2}C_0 \min\{d_+, d_-\}\|\varphi\|, \end{aligned} \tag{D.27}$$

where we have used the inequality (D.18) and $\tilde{H}_\Lambda + \tilde{C}_0 > C_0$. Taking

$$\varphi = e^{-\beta\rho}\frac{1}{H_\Lambda - E - i\varepsilon}e^{\beta\rho}u, \tag{D.28}$$

one has

$$\|e^{-\beta\rho}(H_\Lambda - E - i\varepsilon)^{-1}e^{\beta\rho}u\| \le C_1\|u\|, \tag{D.29}$$

where we have used (D.3), (D.15) and (D.16). Choosing $\Omega = \mathrm{supp}\, v$ in the definition of the distance function $\rho(\mathbf{r})$ and using this bound (D.29), the desired bound (D.25) is obtained as

$$\begin{aligned} \mathrm{Const} \times \|v(H_\Lambda - E - i\varepsilon)^{-1}w\| &\le \|ve^{-\beta\rho}(H_\Lambda - E - i\varepsilon)^{-1}w\| \\ &\le \|v\|_\infty \|e^{-\beta\rho}(H_\Lambda - E - i\varepsilon)^{-1}e^{\beta\rho}e^{-\beta\rho}w\| \\ &\le C_1\|v\|_\infty\|e^{-\beta\rho}w\| \\ &\le \mathrm{Const} \times C_1\|v\|_\infty\|w\|_\infty e^{-\beta r}. \quad \square \end{aligned} \tag{D.30}$$

Next consider the case with $H_\Lambda > E$. Then the Schwarz inequality yields

$$\begin{aligned} \|\varphi\|\|(\tilde{H}_\Lambda - E + i\beta J)\varphi\| &\ge \mathrm{Re}(\varphi, (\tilde{H}_\Lambda - E + i\beta J)\varphi) \\ &= (\varphi, (\tilde{H}_\Lambda - E)\varphi) \\ &\ge [E_0 - E - \hbar^2\beta^2 \||\nabla\rho|\|_\infty^2/(2m_e)]\|\varphi\| \end{aligned} \tag{D.31}$$

for φ in the domain of H_Λ. Here E_0 is the ground state energy of H_Λ. Therefore, in the same way as in the proof of Theorem D.2, one has

$$\begin{aligned} &\left\| v\frac{1}{H_\Lambda - E}w \right\| \le \frac{\mathrm{Const} \times \|v\|_\infty\|w\|_\infty}{E_0 - E - \hbar^2\beta^2\||\nabla\rho|\|_\infty^2/(2m_e)}e^{-\beta r} \\ &\quad \text{for } 0 < \beta < \frac{\sqrt{2m_e(E_0 - E)}}{\hbar\||\nabla\rho|\|_\infty}, \end{aligned} \tag{D.32}$$

where v, w and r are the same as in Theorem D.2.

Finally, we consider the decay of $(H_\Lambda - E - iy)^{-1}$ with $y \neq 0$. To begin with, we note that

$$\begin{aligned}
&\|\varphi\| \|[H_\Lambda - (\hbar^2\beta^2/(2m_e))(\nabla\rho)^2 - E - iy + i\beta J]\varphi\| \\
&\quad \geq |(\varphi, [H_\Lambda - (\hbar^2\beta^2)/(2m_e))(\nabla\rho)^2 - E - iy + i\beta J]\varphi)| \\
&\quad = \sqrt{|(\varphi, [H_\Lambda - (\hbar^2\beta^2/(2m_e))(\nabla\rho)^2 - E]\varphi)|^2 + |(\varphi, (y - \beta J)\varphi)|^2}. \qquad \text{(D.33)}
\end{aligned}$$

Let E' be a real number satisfying

$$E' > E + \frac{\hbar^2\beta^2}{2m_e}\| |\nabla\rho| \|_\infty^2 + C_2 \qquad \text{(D.34)}$$

with a positive constant C_2. Then if the vector φ satisfies $(\varphi, H_\Lambda\varphi) \geq E'\|\varphi\|^2$, the right-hand side of (D.33) is bounded from below by $C_2\|\varphi\|^2$.

Thus it is sufficient to consider the case that $(\varphi, H_\Lambda\varphi) < E'\|\varphi\|^2$. Note that

$$|(\varphi, (y - \beta J)\varphi)| \geq ||y|\|\varphi\|^2 - \beta|(\varphi, J\varphi)||. \qquad \text{(D.35)}$$

The expectation value in the right-hand side is evaluated as

$$\begin{aligned}
|(\varphi, J\varphi)| &\leq \frac{\hbar}{m_e}|(\varphi, \nabla\rho \cdot (\mathbf{p} + e\mathbf{A})\varphi)| \\
&\leq \frac{\hbar}{m_e}\| |\nabla\rho| \|_\infty \|\varphi\| \sqrt{(\varphi, (\mathbf{p} + e\mathbf{A})^2\varphi)} \\
&\leq \hbar\sqrt{\frac{2}{m_e}}\| |\nabla\rho| \|_\infty \|\varphi\| \sqrt{(\varphi, (H_\Lambda + \|V_\Lambda\|_\infty)\varphi)} \\
&\leq \hbar\| |\nabla\rho| \|_\infty \sqrt{2(E' + \|V_\Lambda\|_\infty)/m_e}\|\varphi\|^2. \qquad \text{(D.36)}
\end{aligned}$$

For a given y, we choose a small β and a small C_2 to satisfy

$$|y| > \beta\hbar\| |\nabla\rho| \|_\infty \sqrt{2(E' + \|V_\Lambda\|_\infty)/m_e} + C_2. \qquad \text{(D.37)}$$

Then these bounds yield

$$|(\varphi, (y - \beta J)\varphi)| \geq C_2\|\varphi\|^2. \qquad \text{(D.38)}$$

This implies that the right-hand side of (D.33) is bounded from below by the same $C_2\|\varphi\|^2$. Thus, in both of the cases, one obtains

$$\|[H_\Lambda - (\hbar^2\beta^2/(2m_e))(\nabla\rho)^2 - E - iy + i\beta J]\varphi\| \geq C_2\|\varphi\|. \qquad \text{(D.39)}$$

In the same way as the above, this leads to the decay bound,

$$\|v(H_\Lambda - E - iy)^{-1}w\| \leq \text{Const} \times C_2^{-1}\|v\|_\infty \|w\|_\infty e^{-\beta r}. \qquad \text{(D.40)}$$

Clearly, for a small $|y|$, both of the parameters β and C_2 must be small. But the resolvent always decays exponentially at large distance for any $y \neq 0$.

D.2 The Landau Hamiltonian with $\mathbf{A}_{\mathrm{P}} = 0$

Following [15, 16], we obtain the exponential decay bound (D.85) for the resolvent in Theorem D.7 below.

Consider the two-dimensional single electron in the uniform magnetic field and with a electrostatic potential V_Λ. The Hamiltonian has the form, $H_\Lambda = H_{\mathrm{L}} + V_\Lambda$, on the rectangular box Λ with the periodic boundary conditions as in Sect. 2, where H_{L} is the Landau Hamiltonian (3.1). We also impose the condition of the flux quantization, $|\Lambda|/(2\pi\ell_B^2) \in \mathbf{N}$. We assume $V_\Lambda \in C^2(\Lambda)$. We denote by $Q_{0,\Lambda}^{(n)}$ the spectral projection onto the $n+1$-th Landau level whose energy eigenvalue of H_{L} is given by $\mathcal{E}_n = (n+1/2)\hbar\omega_c$ with $\omega_c = eB/m_e$. We introduce one-parameter families of operators as

$$H_{\mathrm{L}}(\beta) = e^{-\beta\rho} H_{\mathrm{L}} e^{\beta\rho}, \qquad H_\Lambda(\beta) = H_{\mathrm{L}}(\beta) + V_\Lambda, \qquad Q_{0,\Lambda}^{(n)}(\beta) = e^{-\beta\rho} Q_{0,\Lambda}^{(n)} e^{\beta\rho} \tag{D.41}$$

for $\beta \in \mathbf{R}$. Here the distance function $\rho(\mathbf{r})$ is given in the preceding subsection. We write $H_{\mathrm{L}}(\beta) = \tilde{H}_{\mathrm{L}} + i\beta J$, where the operator J is given by (D.4) with $\mathbf{A} = \mathbf{A}_0$, and

$$\tilde{H}_{\mathrm{L}} = H_{\mathrm{L}} - \frac{\hbar^2\beta^2}{2m_e}(\nabla\rho)^2. \tag{D.42}$$

Lemma D.3 *Let z be a complex number satisfying* $\mathrm{dist}(\sigma(\tilde{H}_{\mathrm{L}}), z) \ge \hbar\omega_c/4$, *where $\sigma(\tilde{H}_{\mathrm{L}})$ is the spectrum of the Hamiltonian $\tilde{H}_{\mathrm{L}}$. Then there exists $\kappa(z) \in (0,1)$ which depends only on z such that the following bound is valid*:

$$\|[z - H_{\mathrm{L}}(\beta)]^{-1}\| \le \frac{8}{\hbar\omega_c} \quad \textit{for any real } \beta \textit{ satisfying } |\beta| \le \kappa(z)\ell_B^{-1}. \tag{D.43}$$

Proof Note that, for a vector φ in the domain of the Hamiltonian and for a small β,

$$\begin{aligned}
\|(\tilde{H}_{\mathrm{L}} - z + i\beta J)\varphi\| &= \|[1 + i\beta J(\tilde{H}_{\mathrm{L}} - z)^{-1}](\tilde{H}_{\mathrm{L}} - z)\varphi\| \\
&\ge [1 - |\beta|\|J(\tilde{H}_{\mathrm{L}} - z)^{-1}\|]\|(\tilde{H}_{\mathrm{L}} - z)\varphi\| \\
&\ge [1 - |\beta|\|J(\tilde{H}_{\mathrm{L}} - z)^{-1}\|]\mathrm{dist}(\sigma(\tilde{H}_{\mathrm{L}}), z)\|\varphi\| \\
&\ge \frac{1}{4}\hbar\omega_c[1 - |\beta|\|J(\tilde{H}_{\mathrm{L}} - z)^{-1}\|]\|\varphi\|.
\end{aligned} \tag{D.44}$$

Therefore it is sufficient to show $|\beta|\|J(\tilde{H}_{\mathrm{L}} - z)^{-1}\| < 1/2$ for a small β. Since

$$J = \frac{i\hbar^2}{2m_e}\Delta\rho - \frac{\hbar}{m_e}\nabla\rho \cdot (\mathbf{p} + e\mathbf{A}_0), \tag{D.45}$$

one has

$$\begin{aligned}
\|J(\tilde{H}_{\mathrm{L}} - z)^{-1}\| \le{}& \frac{\hbar^2}{2m_e}\|\Delta\rho\|_\infty \frac{1}{\mathrm{dist}(\sigma(\tilde{H}_{\mathrm{L}}), z)} \\
&+ \frac{2\hbar}{m_e}\||\nabla\rho|\|_\infty \max_s \|(p_s + eA_{0,s})(\tilde{H}_{\mathrm{L}} - z)^{-1}\|.
\end{aligned} \tag{D.46}$$

The norm of the operator in the second term is evaluated as

$$
\begin{aligned}
&(\varphi, (\tilde{H}_{\rm L} - z^*)^{-1}(p_s + eA_{0,s})^2(\tilde{H}_{\rm L} - z)^{-1}\varphi) \\
&\quad \le 2m_e(\varphi, (\tilde{H}_{\rm L} - z^*)^{-1}(\tilde{H}_{\rm L} + \frac{\hbar^2\beta^2}{2m_e}\||\nabla\rho|\|_\infty^2)(\tilde{H}_{\rm L} - z)^{-1}\varphi) \\
&\quad \le 2m_e\left[\frac{1}{\mathrm{dist}(\sigma(\tilde{H}_{\rm L}), z)} + \frac{|z| + \hbar^2\beta^2\||\nabla\rho|\|_\infty^2/(2m_e)}{\mathrm{dist}(\sigma(\tilde{H}_{\rm L}), z)^2}\right]\|\varphi\|^2. \qquad \text{(D.47)}
\end{aligned}
$$

Combining this, (D.46) and the assumption on z, the desired condition for β in (D.43) can be obtained. □

We introduce the integral representation of the projection $Q_{0,\Lambda}^{(n)}$ as

$$
Q_{0,\Lambda}^{(n)} = \frac{1}{2\pi i}\int_\gamma dz' \frac{1}{z' - H_{\rm L}}, \qquad \text{(D.48)}
$$

where the closed path γ encircles the spectrum of the $n+1$-th Landau level. Further we choose the path γ such that the length of the path is bounded as $|\gamma| \le 3\hbar\omega_c$, and that the distance from the spectrum of the two Hamiltonians $H_{\rm L}, \tilde{H}_{\rm L}$ satisfies

$$
\mathrm{dist}(\gamma, \sigma(H_{\rm L})) \ge \hbar\omega_c/4 \quad \text{and} \quad \mathrm{dist}(\gamma, \sigma(\tilde{H}_{\rm L})) \ge \hbar\omega_c/4 \qquad \text{(D.49)}
$$

for any real β satisfying $|\beta|\||\nabla\rho|\|_\infty \le \ell_B^{-1}$. Then, from the above lemma, there exists $\kappa_n \in (0, 1)$ which depends only on the index n such that the representation,

$$
Q_{0,\Lambda}^{(n)}(\beta) = \frac{1}{2\pi i}\int_\gamma dz' \frac{1}{z' - H_{\rm L}(\beta)}, \qquad \text{(D.50)}
$$

is well defined for any β satisfying $|\beta|\||\nabla\rho|\|_\infty \le \kappa_n\ell_B^{-1}$.

Lemma D.4 *Assume the above condition $|\beta|\||\nabla\rho|\|_\infty \le \kappa_n\ell_B^{-1}$. Then the following bound is valid*:

$$
\|[Q_{0,\Lambda}^{(n)}(\beta), V_\Lambda]\| \le C_{0,0}^{(n)}\ell_B \qquad \text{(D.51)}
$$

with

$$
C_{0,0}^{(n)} = \frac{48}{\pi}\left[\ell_B\|\Delta V_\Lambda\|_\infty + 2(4 + \sqrt{8n + 29})\max_{j=x,y}\|\partial_j V_\Lambda\|_\infty\right], \qquad \text{(D.52)}
$$

where we have written $\nabla = (\partial_x, \partial_y)$.

Proof Note that

$$
\begin{aligned}
[Q_{0,\Lambda}^{(n)}(\beta), V_\Lambda] &= \frac{1}{2\pi i}\int_\gamma dz'\left[\frac{1}{z' - H_{\rm L}(\beta)}V_\Lambda - V_\Lambda\frac{1}{z' - H_{\rm L}(\beta)}\right] \\
&= \frac{1}{2\pi i}\int_\gamma dz' \frac{1}{z' - H_{\rm L}(\beta)}[H_{\rm L}(\beta), V_\Lambda]\frac{1}{z' - H_{\rm L}(\beta)}. \qquad \text{(D.53)}
\end{aligned}
$$

The commutator in the right-hand side is computed as

$$
[H_{\rm L}(\beta), V_\Lambda] = -\frac{\hbar^2}{2m_e}\Delta V_\Lambda - \frac{i\hbar}{m_e}\nabla V_\Lambda \cdot \mathbf{\Pi}, \qquad \text{(D.54)}
$$

where we have written $\mathbf{\Pi} = (\Pi_x, \Pi_y) = \mathbf{p} + e\mathbf{A}_0 - i\hbar\beta\nabla\rho$. From these observations, one has

$$\begin{aligned}
&\|[Q_{0,\Lambda}^{(n)}(\beta), V_\Lambda]\| \\
&\quad \le \frac{3\hbar^2\omega_c}{2\pi m_e}\left[\frac{\hbar}{2}\|R(\beta)\|^2\|\Delta V_\Lambda\|_\infty + \|R(\beta)\| \max_{\ell=x,y}\|\partial_\ell V_\Lambda\|_\infty \sum_{j=x,y}\|\Pi_j R(\beta)\|\right] \\
&\quad \le \frac{48}{\pi}\left[\ell_B^2\|\Delta V_\Lambda\|_\infty + \frac{\hbar}{2m_e}\max_{\ell=x,y}\|\partial_\ell V_\Lambda\|_\infty \max_{j=x,y}\|\Pi_j R(\beta)\|\right],
\end{aligned} \tag{D.55}$$

where we have written $R(\beta) = (z' - H_\mathrm{L}(\beta))^{-1}$ for short, and used the bound (D.43). Therefore it is sufficient to estimate $\|\Pi_j R(\beta)\|$. Note that

$$\begin{aligned}
\sum_{j=x,y}\|\Pi_j R(\beta)\varphi\|^2 &= \sum_{j=x,y}(R(\beta)\varphi, \Pi_j^*\Pi_j R(\beta)\varphi) \\
&= 2m_e(R(\beta)\varphi, H_\mathrm{L}(\beta)R(\beta)\varphi) \\
&\quad + 2i\hbar\beta\sum_{j=x,y}(R(\beta)\varphi, (\partial_j\rho)\Pi_j R(\beta)\varphi).
\end{aligned} \tag{D.56}$$

Further

$$\begin{aligned}
\max_{j=x,y}\|\Pi_j R(\beta)\varphi\|^2 &\le 2m_e(\|\varphi\|\|R(\beta)\varphi\| + |z'|\|R(\beta)\varphi\|^2) \\
&\quad + 4\hbar\beta\||\nabla\rho|\|_\infty\|R(\beta)\varphi\|\max_{j=x,y}\|\Pi_j R(\beta)\varphi\| \\
&\le \frac{16m_e^2}{\hbar e B}(8n+13)\|\varphi\|^2 + \frac{32m_e}{\sqrt{\hbar e B}}\|\varphi\|\max_{j=x,y}\|\Pi_j R(\beta)\varphi\|,
\end{aligned} \tag{D.57}$$

where we have used (D.43), $|z'| \le \mathrm{dist}(\mathcal{E}_n, z') + \mathcal{E}_n$ and $\beta\||\nabla\rho|\|_\infty < \ell_B^{-1}$. Solving this, one has

$$\|\Pi_j R(\beta)\| \le \frac{4m_e}{\sqrt{\hbar e B}}(4 + \sqrt{8n+29}) \quad \text{for } j = x, y. \tag{D.58}$$

Substituting this into (D.55), the bound (D.51) is obtained. □

Lemma D.5 *For any given $\epsilon \in (0,1)$, there exists $\kappa_{n,\epsilon} \in (0,1)$ such that $\kappa_{n,\epsilon}$ depends only on ϵ and the index n of the Landau level, and that, for any real β satisfying $|\beta|\||\nabla\rho|\|_\infty \le \kappa_{n,\epsilon}\ell_B^{-1}$, the following bounds are valid*:

$$\|Q_{0,\Lambda}^{(n)}(\beta) - Q_{0,\Lambda}^{(n)}\| \le \epsilon \tag{D.59}$$

and

$$\begin{aligned}
&\|(H_\mathrm{L}(\beta) - z)[Q_{0,\Lambda}^{(n)}(\beta) - Q_{0,\Lambda}^{(n)}]\| \le \epsilon\hbar\omega_c \\
&\quad \textit{for } z \textit{ satisfying } \mathrm{dist}(z, \mathcal{E}_n) \le \Delta\mathcal{E}_{\max}
\end{aligned} \tag{D.60}$$

with a positive constant $\Delta\mathcal{E}_{\max}$.

Proof Note that

$$\begin{aligned}
Q_{0,\Lambda}^{(n)}(\beta) - Q_{0,\Lambda}^{(n)} &= \frac{1}{2\pi i}\int_\gamma dz' \left[\frac{1}{z' - H_{\mathrm{L}}(\beta)} - \frac{1}{z' - H_{\mathrm{L}}}\right] \\
&= \frac{1}{2\pi i}\int_\gamma dz' \frac{1}{z' - H_{\mathrm{L}}(\beta)}(H_{\mathrm{L}}(\beta) - H_{\mathrm{L}})\frac{1}{z' - H_{\mathrm{L}}} \\
&= \frac{1}{2\pi i}\int_\gamma dz' \frac{1}{z' - H_{\mathrm{L}}(\beta)}\left[-\frac{\hbar^2\beta^2}{2m_e}(\nabla\rho)^2 + i\beta J\right]\frac{1}{z' - H_{\mathrm{L}}} \qquad \text{(D.61)}
\end{aligned}$$

for any β satisfying $|\beta| \le \kappa_n \ell_B^{-1}$. Therefore the norm of the left-hand side is evaluated as

$$\begin{aligned}
&\|Q_{0,\Lambda}^{(n)}(\beta) - Q_{0,\Lambda}^{(n)}\| \\
&\quad \le \frac{3\hbar\omega_c}{2\pi}\sup_{z'\in\Gamma}\left\|\frac{1}{z' - H_{\mathrm{L}}(\beta)}\right\|\left(\frac{\hbar^2\beta^2}{2m_e}\||\nabla\rho|\|_\infty^2\left\|\frac{1}{z' - H_{\mathrm{L}}}\right\| + \beta\left\|J\frac{1}{z' - H_{\mathrm{L}}}\right\|\right) \\
&\quad \le \frac{12}{\pi}(2\ell_B^2\beta^2\||\nabla\rho|\|_\infty^2 + \epsilon'), \qquad \text{(D.62)}
\end{aligned}$$

where we have used the bound (D.43) and $\|(z' - H_{\mathrm{L}})^{-1}\| \le 4/(\hbar\omega_c)$, and we have chosen β so that, for a given small ϵ',

$$|\beta|\left\|J\frac{1}{z' - H_{\mathrm{L}}}\right\| \le \epsilon' \qquad \text{(D.63)}$$

which can be proved in the same way as in Lemma D.3. The resulting bound (D.62) with a small β implies the desired bound (D.59).

In order to obtain the second bound, we note that

$$\begin{aligned}
&(H_{\mathrm{L}}(\beta) - z)[Q_{0,\Lambda}^{(n)}(\beta) - Q_{0,\Lambda}^{(n)}] \\
&\quad = \frac{1}{2\pi i}\int_\gamma dz'\left[-1 + \frac{z' - z}{z' - H_{\mathrm{L}}(\beta)}\right](H_{\mathrm{L}}(\beta) - H_{\mathrm{L}})\frac{1}{z' - H_{\mathrm{L}}}. \qquad \text{(D.64)}
\end{aligned}$$

In the same way as in the above, the norm is estimated as

$$\begin{aligned}
&\|(H_{\mathrm{L}}(\beta) - z)[Q_{0,\Lambda}^{(n)}(\beta) - Q_{0,\Lambda}^{(n)}]\| \\
&\quad \le \frac{3\hbar\omega_c}{2\pi}\left(9 + 8\frac{\Delta\mathcal{E}_{\max}}{\hbar\omega_c}\right)\left\|(H_{\mathrm{L}}(\beta) - H_{\mathrm{L}})\frac{1}{z' - H_{\mathrm{L}}}\right\|. \qquad \text{(D.65)}
\end{aligned}$$

Here we have used $|z' - z| \le \mathrm{dist}(\mathcal{E}_n, z') + \mathrm{dist}(\mathcal{E}_n, z) \le \hbar\omega_c + \Delta\mathcal{E}_{\max}$. The norm of the operator in the right-hand side is already estimated in the above. □

We write $z = E + i\varepsilon$ with $E, \varepsilon \in \mathbf{R}$.

Lemma D.6 *Suppose* $\mathcal{E}_{n-1} + \|V_\Lambda^+\|_\infty + \hat{\delta}_-\hbar\omega_c \le E \le \mathcal{E}_{n+1} - \|V_\Lambda^-\|_\infty - 2\hat{\delta}_+\hbar\omega_c$ *for* $n = 0, 1, 2, \ldots$ *with some positive constants* $\hat{\delta}_\pm$ *and with* $\mathcal{E}_{-1} = -\infty$. *Let* φ *be a vector in the domain of the Hamiltonian. Then the following bound is valid*:

$$\|(H_\Lambda(\beta) - z)[1 - Q_{0,\Lambda}^{(n)}]\varphi\| \ge C_{0,1}^{(n)}\hbar\omega_c\|[1 - Q_{0,\Lambda}^{(n)}]\varphi\| \qquad \text{(D.66)}$$

for any β *satisfying* $|\beta|\||\nabla\rho|\|_\infty \le \kappa_n'\ell_B^{-1}$, *where*

$$C_{0,1}^{(n)} = \begin{cases} \hat{\delta}_+, & \text{for } n=0; \\ \min\{\hat{\delta}_+, \hat{\delta}_-\}/2, & \text{for } n=1,2,\ldots, \end{cases} \tag{D.67}$$

and

$$\kappa_n' = \begin{cases} \sqrt{2\hat{\delta}_+}, & \text{for } n=0; \\ \min\left\{\frac{\sqrt{\hat{\delta}_+\hat{\delta}_-}}{\ell_B\|\Delta\rho\|_\infty/\||\nabla\rho|\|_\infty + 2\sqrt{2n-1}}, \sqrt{2\hat{\delta}_+}\right\}, & \text{for } n=1,2,\ldots. \end{cases} \tag{D.68}$$

Proof We write $\psi = [1 - Q_{0,\Lambda}^{(n)}]\varphi$, and decompose the vector ψ into two parts as

$$\varphi_+ = \sum_{j>n} Q_{0,\Lambda}^{(j)}\varphi \quad \text{and} \quad \varphi_- = \sum_{j<n} Q_{0,\Lambda}^{(j)}\varphi. \tag{D.69}$$

Then one has

$$\begin{aligned} &\|\psi\|\|(H_\Lambda(\beta) - z)\psi\| \\ &\ge \mathrm{Re}\left(\varphi_+ - \varphi_-, \left(H_\Lambda - E - \frac{\hbar^2\beta^2}{2m_e}(\nabla\rho)^2 - i\varepsilon + i\beta J\right)\varphi_+ + \varphi_-\right) \\ &\ge \hbar\omega_c\hat{\delta}_+\|\varphi_+\|^2 + \hbar\omega_c\hat{\delta}_-\|\varphi_-\|^2 - 2\beta\,\mathrm{Im}(\varphi_+, J\varphi_-), \end{aligned} \tag{D.70}$$

where we have used

$$\mathcal{E}_{n+1} - \|V_\Lambda^-\|_\infty - E - \hbar^2\beta^2\||\nabla\rho|\|_\infty^2/(2m_e) \ge \hat{\delta}_+\hbar\omega_c \tag{D.71}$$

and

$$E - \mathcal{E}_{n-1} - \|V_\Lambda^+\|_\infty \ge \hat{\delta}_-\hbar\omega_c \tag{D.72}$$

which are easily derived from the assumptions. Clearly, in the case with $n=0$, one has $\varphi_- = 0$, and so the statement holds. For the rest of the cases, we use the following bound:

$$\begin{aligned} |(\varphi_+, J\varphi_-)| &\le \frac{\hbar^2}{2m_e}\|\Delta\rho\|_\infty\|\varphi_+\|\|\varphi_-\| + \frac{\hbar}{m_e}|(\varphi_+, \nabla\rho\cdot(\mathbf{p}+e\mathbf{A}_0)\varphi_-)| \\ &\le \frac{\hbar^2}{2m_e}\|\Delta\rho\|_\infty\|\varphi_+\|\|\varphi_-\| + \frac{\hbar}{m_e}\sqrt{(\varphi_+, |\nabla\rho|^2\varphi_+)(\varphi_-, (\mathbf{p}+e\mathbf{A}_0)^2\varphi_-)} \\ &\le \left(\frac{\hbar^2}{2m_e}\|\Delta\rho\|_\infty + \hbar\sqrt{\frac{2\mathcal{E}_{n-1}}{m_e}}\||\nabla\rho|\|_\infty\right)\|\varphi_+\|\|\varphi_-\|, \end{aligned} \tag{D.73}$$

where we have used (D.45) and the Schwarz inequality. Combining this, (D.70) and the assumption on β, the desired bound is obtained. □

Let us estimate $\|(H_\Lambda(\beta) - z)\varphi\|$ for a vector φ in the domain of the Hamiltonian and for $z \in \mathbf{C}$. We take β satisfying $|\beta|\||\nabla\rho|\|_\infty \le \ell_B^{-1}\min\{\kappa_{n,\epsilon}, \kappa_n'\}$ for a given ϵ. Note that

$$\begin{aligned} \|Q_{0,\Lambda}^{(n)}(H_\Lambda(\beta) - z)\varphi\| &\ge \|e^{-\beta\rho}(H_\Lambda - z)Q_{0,\Lambda}^{(n)}e^{\beta\rho}\varphi\| \\ &\quad - \|Q_{0,\Lambda}^{(n)}(H_\Lambda(\beta) - z)\varphi - e^{-\beta\rho}(H_\Lambda - z)Q_{0,\Lambda}^{(n)}e^{\beta\rho}\varphi\|. \end{aligned} \tag{D.74}$$

The first term in the right-hand side can be evaluated as

$$\begin{aligned}\|e^{-\beta\rho}(H_\Lambda - z)Q_{0,\Lambda}^{(n)}e^{\beta\rho}\varphi\| &= \|(\mathcal{E}_n + V_\Lambda - z)e^{-\beta\rho}Q_{0,\Lambda}^{(n)}e^{\beta\rho}\varphi\|\\ &\geq \Delta E\|Q_{0,\Lambda}^{(n)}(\beta)\varphi\|\\ &\geq \Delta E\|Q_{0,\Lambda}^{(n)}\varphi\| - \Delta E\|[Q_{0,\Lambda}^{(n)}(\beta) - Q_{0,\Lambda}^{(n)}]\varphi\|\\ &\geq \Delta E\|Q_{0,\Lambda}^{(n)}\varphi\| - \epsilon\Delta E\|\varphi\|,\end{aligned}\tag{D.75}$$

where

$$\Delta E = \inf|\mathcal{E}_n + V_\Lambda - \mathrm{Re}\, z|\tag{D.76}$$

and we have used the inequality (D.59). The second term in the right-hand side of (D.74) can be evaluated as

$$\begin{aligned}&\|Q_{0,\Lambda}^{(n)}(H_\Lambda(\beta) - z)\varphi - e^{-\beta\rho}(H_\Lambda - z)Q_{0,\Lambda}^{(n)}e^{\beta\rho}\varphi\|\\ &\quad= \|Q_{0,\Lambda}^{(n)}(H_\Lambda(\beta) - z)\varphi - (H_\Lambda(\beta) - z)Q_{0,\Lambda}^{(n)}(\beta)\varphi\|\\ &\quad\leq \|[Q_{0,\Lambda}^{(n)} - Q_{0,\Lambda}^{(n)}(\beta)](H_\Lambda(\beta) - z)\varphi\| + \|[[Q_{0,\Lambda}^{(n)}(\beta), H_\Lambda(\beta)]\varphi\|\\ &\quad\leq \|[Q_{0,\Lambda}^{(n)} - Q_{0,\Lambda}^{(n)}(\beta)](H_\Lambda(\beta) - z)\varphi\| + \|[[Q_{0,\Lambda}^{(n)}(\beta), V_\Lambda]\varphi\|\\ &\quad\leq \epsilon\|(H_\Lambda(\beta) - z)\varphi\| + C_{0,0}^{(n)}\ell_B\|\varphi\|,\end{aligned}\tag{D.77}$$

we have used the inequalities (D.51) and (D.59). Substituting these bounds into (D.74), one has

$$(1+\epsilon)\|(H_\Lambda(\beta) - z)\varphi\| \geq \Delta E\|Q_{0,\Lambda}^{(n)}\varphi\| - (\epsilon\Delta E + C_{0,0}^{(n)}\ell_B)\|\varphi\|.\tag{D.78}$$

Note that

$$\begin{aligned}&\|(1 - Q_{0,\Lambda}^{(n)})(H_\Lambda(\beta) - z)\varphi\|\\ &\quad\geq \|[1 - Q_{0,\Lambda}^{(n)}(\beta)](H_\Lambda(\beta) - z)\varphi\| - \|[Q_{0,\Lambda}^{(n)}(\beta) - Q_{0,\Lambda}^{(n)}](H_\Lambda(\beta) - z)\varphi\|\\ &\quad\geq \|(H_\Lambda(\beta) - z)[1 - Q_{0,\Lambda}^{(n)}(\beta)]\varphi\|\\ &\qquad - \|[Q_{0,\Lambda}^{(n)}(\beta), H_\Lambda(\beta)]\varphi\| - \|[Q_{0,\Lambda}^{(n)}(\beta) - Q_{0,\Lambda}^{(n)}](H_\Lambda(\beta) - z)\varphi\|\\ &\quad\geq \|(H_\Lambda(\beta) - z)[1 - Q_{0,\Lambda}^{(n)}]\varphi\|\\ &\qquad - \|(H_\Lambda(\beta) - z)[Q_{0,\Lambda}^{(n)}(\beta) - Q_{0,\Lambda}^{(n)}]\varphi\| - \|[Q_{0,\Lambda}^{(n)}(\beta), V_\Lambda]\varphi\|\\ &\qquad - \|[Q_{0,\Lambda}^{(n)}(\beta) - Q_{0,\Lambda}^{(n)}](H_\Lambda(\beta) - z)\varphi\|.\end{aligned}\tag{D.79}$$

Using the inequalities (D.51), (D.59), (D.60) and (D.66) for this right-hand side, one has

$$\begin{aligned}&(1+\epsilon)\|(H_\Lambda(\beta) - z)\varphi\|\\ &\quad\geq C_{0,1}^{(n)}\hbar\omega_c\|[1 - Q_{0,\Lambda}^{(n)}]\varphi\| - (\epsilon\hbar\omega_c + \epsilon\|V_\Lambda\|_\infty + C_{0,0}^{(n)}\ell_B)\|\varphi\|.\end{aligned}\tag{D.80}$$

Combining this with the above inequality (D.78), one has

$$
\begin{aligned}
&(1+\epsilon)\left(\frac{1}{\Delta E}+\frac{1}{C_{0,1}^{(n)}\hbar\omega_c}\right)\|(H_\Lambda(\beta)-z)\varphi\| \\
&\quad\geq\left\{1-\epsilon\left[1+\frac{1}{C_{0,1}^{(n)}}(1+\frac{\|V_\Lambda\|_\infty}{\hbar\omega_c})\right]-C_{0,0}^{(n)}\ell_B\left(\frac{1}{\Delta E}+\frac{1}{C_{0,1}^{(n)}\hbar\omega_c}\right)\right\}\|\varphi\|. \quad \text{(D.81)}
\end{aligned}
$$

Assume that the energy E satisfies the condition in Lemma D.6 with the positive constants $\hat{\delta}_\pm$ which are independent of the strength B of the uniform magnetic field. Under this assumption, $C_{0,1}^{(n)}$ and κ_n' in Lemma D.6 can be chosen to be independent of B except for a small B. Further assume that ΔE satisfies $\Delta E \geq \Delta\mathcal{E} > 0$ with some constant $\Delta\mathcal{E}$ which is independent of B. Then there exists $B_{0,1}^{(n)}$ such that, for any $B > B_{0,1}^{(n)}$,

$$
C_{0,0}^{(n)}\ell_B[(\Delta E)^{-1}+(C_{0,1}^{(n)}\hbar\omega_c)^{-1}]\leq 1/3. \quad \text{(D.82)}
$$

Moreover we can choose ϵ satisfying

$$
\epsilon\{1+(C_{0,1}^{(n)})^{-1}[1+\|V_\Lambda\|_\infty/(\hbar\omega_c)]\}\leq 1/3. \quad \text{(D.83)}
$$

Substituting these into the above bound (D.81), one has

$$
C_{0,2}^{(n)}(\Delta\mathcal{E})^{-1}\|(H_\Lambda(\beta)-z)\varphi\|\geq\|\varphi\| \quad \text{for any } B\geq B_{0,1}^{(n)}, \quad \text{(D.84)}
$$

where the positive constant $C_{0,2}^{(n)}$ depends only on the index n. Therefore a similar decay estimate for the resolvent is obtained in the same way as in Theorem D.2. We summarize the result as

Theorem D.7 *Let v, w be bounded functions with a compact support, and suppose that the boundary of the region* $\operatorname{supp} v$ *is smooth. Write $z = E + i\varepsilon$ with $E, \varepsilon \in \mathbf{R}$. Assume that the energy E satisfies the condition in Lemma D.6 with the positive constants $\hat{\delta}_\pm$ which are independent of the strength B of the uniform magnetic field. Further assume that ΔE of (D.76) satisfies $\Delta E \geq \Delta\mathcal{E} > 0$ with some constant $\Delta\mathcal{E}$ which is independent of B. Then there exist $B_{0,1}^{(n)}$ and $\tilde{\kappa}_n$ which depend only on the index n of the Landau level such that*

$$
\|v(H_\Lambda-E-i\varepsilon)^{-1}w\|\leq\frac{C_{0,3}^{(n)}}{\Delta\mathcal{E}}\|v\|_\infty\|w\|_\infty\exp[-\tilde{\kappa}_n\ell_B^{-1}r] \quad \textit{for any } B\geq B_{0,1}^{(n)}. \quad \text{(D.85)}
$$

Here $r = \operatorname{dist}(\operatorname{supp} v, \operatorname{supp} w)$ and the positive constant $C_{0,3}^{(n)}$ depends only on the index n.

Appendix E: Proof of Lemma 5.1

The first inequality (5.18) can be obtained as

$$
\begin{aligned}
&\|\alpha_i(p_i+eA_i)R\psi\|^2 \\
&\quad=(\psi, R^*(p_i+eA_i)|\alpha_i|^2(p_i+eA_i)R\psi) \\
&\quad\leq\|\alpha_i\|_\infty^2(\psi, R^*(p_i+eA_i)^2R\psi) \\
&\quad\leq 2m_e\|\alpha_i\|_\infty^2\{\|R\|+[|E|+\|(V_0^-+V_\omega^-)\|_\infty]\|R\|^2\}\|\psi\|^2 \quad \text{(E.1)}
\end{aligned}
$$

for $i = x, y$ and for any vector ψ. Here we have used

$$\sum_{i=x,y} R^*(p_i + eA_i)^2 R \le 2m_e R^*[H_\omega + \|(V_0^- + V_\omega^-)\|_\infty]R$$
$$\le m_e\{(R + R^*) + 2[|E| + \|(V_0^- + V_\omega^-)\|_\infty]R^*R\}. \qquad \text{(E.2)}$$

In order to get the second inequality, we first note that

$$\begin{aligned}&\|(p_i + eA_i)R(\mathbf{p} + e\mathbf{A}) \cdot \boldsymbol{\alpha}\varphi\|^2 \\ &\quad = (\varphi, \boldsymbol{\alpha} \cdot (\mathbf{p} + e\mathbf{A})R^*(p_i + eA_i)^2 R(\mathbf{p} + e\mathbf{A}) \cdot \boldsymbol{\alpha}\varphi) \\ &\quad \le m_e(\varphi, \boldsymbol{\alpha} \cdot (\mathbf{p} + e\mathbf{A})(R + R^*)(\mathbf{p} + e\mathbf{A}) \cdot \boldsymbol{\alpha}\varphi) \\ &\qquad + 2m_e[|E| + \|(V_0^- + V_\omega^-)\|_\infty](\varphi, \boldsymbol{\alpha} \cdot (\mathbf{p} + e\mathbf{A})R^*R(\mathbf{p} + e\mathbf{A}) \cdot \boldsymbol{\alpha}\varphi). \qquad \text{(E.3)}\end{aligned}$$

Using the Schwarz inequality, one has

$$\begin{aligned}&(\varphi, \boldsymbol{\alpha} \cdot (\mathbf{p} + e\mathbf{A})R^*R(\mathbf{p} + e\mathbf{A}) \cdot \boldsymbol{\alpha}\varphi) \\ &\quad \le \||\boldsymbol{\alpha}|\|_\infty \|\varphi\| \sqrt{(\varphi, \boldsymbol{\alpha} \cdot (\mathbf{p} + e\mathbf{A})R^*R(\mathbf{p} + e\mathbf{A})^2 R^*R(\mathbf{p} + e\mathbf{A}) \cdot \boldsymbol{\alpha}\varphi)}. \qquad \text{(E.4)}\end{aligned}$$

Combining this with the inequality (E.2), one obtains the bound (5.20). Similarly,

$$\begin{aligned}&|(\varphi, \boldsymbol{\alpha} \cdot (\mathbf{p} + e\mathbf{A})R(\mathbf{p} + e\mathbf{A}) \cdot \boldsymbol{\alpha}\varphi)| \\ &\quad \le \||\boldsymbol{\alpha}|\|_\infty \|\varphi\| \sqrt{(\varphi, \boldsymbol{\alpha} \cdot (\mathbf{p} + e\mathbf{A})R^*(\mathbf{p} + e\mathbf{A})^2 R(\mathbf{p} + e\mathbf{A}) \cdot \boldsymbol{\alpha}\varphi)}. \qquad \text{(E.5)}\end{aligned}$$

Combining this with the inequality (E.2), one obtains

$$\begin{aligned}&|(\varphi, \boldsymbol{\alpha} \cdot (\mathbf{p} + e\mathbf{A})R(\mathbf{p} + e\mathbf{A}) \cdot \boldsymbol{\alpha}\varphi)|^2 \\ &\quad \le m_e \||\boldsymbol{\alpha}|\|_\infty^2 \|\varphi\|^2 \{|(\varphi, \boldsymbol{\alpha} \cdot (\mathbf{p} + e\mathbf{A})R(\mathbf{p} + e\mathbf{A}) \cdot \boldsymbol{\alpha}\varphi)| \\ &\qquad + [|E| + \|(V_0^- + V_\omega^-)\|_\infty](\varphi, \boldsymbol{\alpha} \cdot (\mathbf{p} + e\mathbf{A})R^*R(\mathbf{p} + e\mathbf{A}) \cdot \boldsymbol{\alpha}\varphi)\} \\ &\quad \le 2m_e \||\boldsymbol{\alpha}|\|_\infty^2 \|\varphi\|^2 |(\varphi, \boldsymbol{\alpha} \cdot (\mathbf{p} + e\mathbf{A})R(\mathbf{p} + e\mathbf{A}) \cdot \boldsymbol{\alpha}\varphi)| \\ &\qquad + 4m_e^2 \||\boldsymbol{\alpha}|\|_\infty^4 f_{E,R}(1 + f_{E,R})\|\varphi\|^4, \qquad \text{(E.6)}\end{aligned}$$

where we have used the bound (5.20). Solving this quadratic inequality, one has

$$|(\varphi, \boldsymbol{\alpha} \cdot (\mathbf{p} + e\mathbf{A})R(\mathbf{p} + e\mathbf{A}) \cdot \boldsymbol{\alpha}\varphi)| \le 2m_e \||\boldsymbol{\alpha}|\|_\infty^2 (1 + f_{E,R})\|\varphi\|^2. \qquad \text{(E.7)}$$

Substituting this and (5.20) into the right-hand side of (E.3), the desired bound (5.19) is obtained.

Appendix F: Proofs of Lemmas 6.2 and 6.3

For the purpose of this appendix, we prepare the following three lemmas:

Lemma F.1

(i) *Let ℓ, ℓ' be odd integers larger than 1 such that ℓ' is a multiple of ℓ. Let A^{good} be the event that no two disjoint γ-bad boxes of size 3ℓ with center in $\Gamma_\ell \cap \Lambda_{5\ell'}(\mathbf{z})$ exist.*

Assume $\mathrm{Prob}[\Lambda_{3\ell}(\cdots)$ *is γ-good*$] \ge 1-\eta$ *with a small* $\eta > 0$. *Then* $\mathrm{Prob}(A^{\mathrm{good}}) \ge 1 - (5\ell'/\ell)^4\eta^2$.

(ii) *Assume that the event* A^{good} *occurs. Let* $\mathbf{u}, \mathbf{v} \in \Gamma_\ell$ *such that* $\Lambda_\ell(\mathbf{u}) \subset \Lambda_{\ell'}(\mathbf{z})$ *and that* $\Lambda_\ell(\mathbf{v}) \cap (\Lambda_{3\ell'}(\mathbf{z}) \backslash \Lambda_{3\ell'}^{\delta}(\mathbf{z})) \ne \emptyset$. *Then*

$$\|\chi_\ell(\mathbf{u}) R_{5\ell',\mathbf{z}}(E+i\varepsilon)\chi_\ell(\mathbf{v})\| \le (8e^{-\gamma\ell})^{\ell'/\ell-4}\|R_{5\ell',\mathbf{z}}(E+i\varepsilon)\|. \tag{F.1}$$

Proof The statement (i) follows from elementary combinatorics.

(ii) Using the geometric resolvent equation,

$$\chi_{3\ell}^{\delta}(\mathbf{u})R_{5\ell',\mathbf{z}} = R_{3\ell,\mathbf{u}}\chi_{3\ell}^{\delta}(\mathbf{u}) + R_{3\ell,\mathbf{u}}\tilde{W}_{3\ell}^{\delta}(\mathbf{u})R_{5\ell',\mathbf{z}}, \tag{F.2}$$

one has

$$\begin{aligned}\chi_\ell(\mathbf{u})R_{5\ell',\mathbf{z}}\chi_\ell(\mathbf{v}) &= \chi_\ell(\mathbf{u})\chi_{3\ell}^{\delta}(\mathbf{u})R_{5\ell',\mathbf{z}}\chi_\ell(\mathbf{v})\\ &= \chi_\ell(\mathbf{u})R_{3\ell,\mathbf{u}}\tilde{W}_{3\ell}^{\delta}(\mathbf{u})R_{5\ell',\mathbf{z}}\chi_\ell(\mathbf{v})\\ &= \chi_\ell(\mathbf{u})R_{3\ell,\mathbf{u}}\tilde{W}_{3\ell}^{\delta}(\mathbf{u})\sum_{\tilde{\mathbf{u}}\in\Gamma_\ell\cap(\Lambda_{3\ell}(\mathbf{u})\backslash\Lambda_\ell(\mathbf{u}))}\chi_\ell(\tilde{\mathbf{u}})R_{5\ell',\mathbf{z}}\chi_\ell(\mathbf{v}),\end{aligned} \tag{F.3}$$

where we have written $R_{\ell,\mathbf{z}}$ for $R_{\ell,\mathbf{z}}(E+i\varepsilon)$ for short. We can choose $\mathbf{u}_1$ from the set of $\tilde{\mathbf{u}}$ so that $\|\chi_\ell(\tilde{\mathbf{u}})R_{5\ell',\mathbf{z}}\chi_\ell(\mathbf{v})\|$ becomes maximal. Thus one has

$$\|\chi_\ell(\mathbf{u})R_{5\ell',\mathbf{z}}\chi_\ell(\mathbf{v})\| \le 8e^{-\gamma\ell}\|\chi_\ell(\mathbf{u}_1)R_{5\ell',\mathbf{z}}\chi_\ell(\mathbf{v})\| \tag{F.4}$$

when $\Lambda_{3\ell}(\mathbf{u})$ is γ-good. Since the norm of the operator in the right-hand side can be estimated in the same way, one can repeat this procedure and construct the points, $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k \in \Gamma_\ell$, as long as $\Lambda_{3\ell}(\mathbf{u}_{k-1})$ is γ-good and does not hit $\Lambda_\ell(\mathbf{v})$ or $\partial\Lambda_{5\ell'}(\mathbf{z})$.

The same type of estimate can be applied to $\mathbf{v}$ as a starting point as follows: Using the adjoint of the geometric resolvent equation,

$$R_{5\ell',\mathbf{z}}\chi_{3\ell}^{\delta}(\mathbf{v}) = \chi_{3\ell}^{\delta}(\mathbf{v})R_{3\ell,\mathbf{v}} + R_{5\ell',\mathbf{z}}(\tilde{W}_{3\ell}^{\delta}(\mathbf{v}))^* R_{3\ell,\mathbf{v}}, \tag{F.5}$$

one has

$$\begin{aligned}\chi_\ell(\mathbf{u}_k)R_{5\ell',\mathbf{z}}\chi_\ell(\mathbf{v}) &= \chi_\ell(\mathbf{u}_k)R_{5\ell',\mathbf{z}}\chi_{3\ell}^{\delta}(\mathbf{v})\chi_\ell(\mathbf{v})\\ &= \chi_\ell(\mathbf{u}_k)R_{5\ell',\mathbf{z}}(\tilde{W}_{3\ell}^{\delta}(\mathbf{v}))^* R_{3\ell,\mathbf{v}}\chi_\ell(\mathbf{v})\\ &= \chi_\ell(\mathbf{u}_k)R_{5\ell',\mathbf{z}}\sum_{\tilde{\mathbf{v}}\in\Gamma_\ell\cap(\Lambda_{3\ell}(\mathbf{v})\backslash\Lambda_\ell(\mathbf{v}))}\chi_\ell(\tilde{\mathbf{v}})(\tilde{W}_{3\ell}^{\delta}(\mathbf{v}))^* R_{3\ell,\mathbf{v}}\chi_\ell(\mathbf{v}).\end{aligned} \tag{F.6}$$

Thus

$$\|\chi_\ell(\mathbf{u}_k)R_{5\ell',\mathbf{z}}\chi_\ell(\mathbf{v})\| \le 8e^{-\gamma\ell}\|\chi_\ell(\mathbf{u}_k)R_{5\ell',\mathbf{z}}\chi_\ell(\mathbf{v}_1)\| \tag{F.7}$$

when $\Lambda_{3\ell}(\mathbf{v})$ is γ-good. In the same way, the procedure yields the points, $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_j$, and one obtains the bound,

$$\|\chi_\ell(\mathbf{u})R_{5\ell',\mathbf{z}}\chi_\ell(\mathbf{v})\| \le (8e^{-\gamma\ell})^{k+j}\|\chi_\ell(\mathbf{u}_k)R_{5\ell',\mathbf{z}}\chi_\ell(\mathbf{v}_j)\| \le (8e^{-\gamma\ell})^{k+j}\|R_{5\ell',\mathbf{z}}\|. \tag{F.8}$$

This process moves in steps of ℓ. The assumption that A^{good} occurs implies that there may be only one cluster of overlapping γ-bad boxes. The diameter of such a bad cluster is at most

5ℓ. From these observations, one has that $k+j \ge |\mathbf{u}-\mathbf{v}|/\ell - 4$ iterations can be performed before the process stops on the both sides. Since $|\mathbf{u}-\mathbf{v}| \ge \ell'$ from the assumption, the desired bound (F.1) is obtained. When the process hits the boundary of $\Lambda_{5\ell'}(\mathbf{z})$ without hitting a γ-bad box, we have $\ell k \ge 2\ell'$ or $\ell j \ge \ell'$. Therefore the bound (F.1) remains valid. □

Lemma F.2 *Let ℓ, ℓ' be odd integers larger than* 1 *such that ℓ' is a multiple of ℓ and satisfies $\ell' > 4\ell$. Assume that the event A^{good} given in the preceding lemma occurs. Then*

$$\begin{aligned}&\|\chi_{\ell'}(\mathbf{z})R_{3\ell',\mathbf{z}}(\tilde{W}^{\delta}_{3\ell'}(\mathbf{z}))^{*}\|\\&\quad\le 6\left(\frac{\ell'}{\ell}\right)^{3}(8e^{-\gamma\ell})^{\ell'/\ell-4}[f_4(|E|,\|R_{5\ell',\mathbf{z}}\|)+f_5(|E|,\|R_{3\ell',\mathbf{z}}\|)]\|R_{5\ell',\mathbf{z}}\|, \qquad \text{(F.9)}\end{aligned}$$

where the functions, f_4 and f_5, are given by

$$\begin{aligned}f_4(|E|,\|R\|) &= 2C_{\delta,\omega} + \frac{16\hbar^2}{m_e}\max_{m,n=x,y}\{\|\partial_m\phi_{n,3\ell'}(\mathbf{z})\|_\infty\}(1+f_{E,R})\\&\quad + 2\sqrt{2}\left(\frac{\hbar}{\sqrt{m_e}}\right)^{3}\max_{n=x,y}\{\|\Delta\phi_{n,3\ell'}(\mathbf{z})\|_\infty\}(1+f_{E,R})^{1/2}\|R\|^{1/2} \qquad \text{(F.10)}\end{aligned}$$

and

$$\begin{aligned}f_5(|E|,\|R\|) &= C^2_{\delta,\omega}\|R\| + \frac{2\sqrt{2}\hbar}{\sqrt{m_e}}C_{\delta,\omega}[\||\nabla\chi^{\delta}_{3\ell'}(\mathbf{z})|\|_\infty\\&\quad + 2\max_{m=x,y}\{\|\partial_m\chi^{\delta}_{3\ell'}(\mathbf{z})\|_\infty\}](1+f_{E,R})^{1/2}\|R\|^{1/2}\\&\quad + \frac{4\hbar^2}{m_e}\max_{m=x,y}\{\|\partial_m\chi^{\delta}_{3\ell'}(\mathbf{z})\|_\infty\}\||\nabla\chi^{\delta}_{3\ell'}(\mathbf{z})|\|_\infty(1+f_{E,R}). \qquad \text{(F.11)}\end{aligned}$$

Here

$$C_{\delta,\omega} = \frac{\hbar^2}{2m_e}\|\Delta\chi^{\delta}_{3\ell'}(\mathbf{z})\|_\infty + \|V_\omega\|_\infty, \qquad \phi_{i,3\ell'}(\mathbf{z}) = \chi^{\delta}_{3\ell'}(\mathbf{z})\partial_i\chi^{\delta}_{3\ell'}(\mathbf{z}), \qquad \text{(F.12)}$$

and the function $f_{E,R}$ is given by (5.21).

Proof Using the adjoint of the geometric resolvent equation (F.5), one has

$$\begin{aligned}&\chi_{\ell'}(\mathbf{z})R_{5\ell',\mathbf{z}}\chi^{\delta}_{3\ell'}(\mathbf{z})(\tilde{W}^{\delta}_{3\ell'}(\mathbf{z}))^{*}\\&\quad= \chi_{\ell'}(\mathbf{z})R_{3\ell',\mathbf{z}}(\tilde{W}^{\delta}_{3\ell'}(\mathbf{z}))^{*} + \chi_{\ell'}(\mathbf{z})R_{5\ell',\mathbf{z}}(\tilde{W}^{\delta}_{3\ell'}(\mathbf{z}))^{*}R_{3\ell',\mathbf{z}}(\tilde{W}^{\delta}_{3\ell'}(\mathbf{z}))^{*}. \qquad \text{(F.13)}\end{aligned}$$

Therefore

$$\begin{aligned}&\|\chi_{\ell'}(\mathbf{z})R_{3\ell',\mathbf{z}}(\tilde{W}^{\delta}_{3\ell'}(\mathbf{z}))^{*}\varphi\|\\&\quad\le \|\chi_{\ell'}(\mathbf{z})R_{5\ell',\mathbf{z}}\chi^{\delta}_{3\ell'}(\mathbf{z})(\tilde{W}^{\delta}_{3\ell'}(\mathbf{z}))^{*}\varphi\|\\&\qquad + \|\chi_{\ell'}(\mathbf{z})R_{5\ell',\mathbf{z}}(\tilde{W}^{\delta}_{3\ell'}(\mathbf{z}))^{*}R_{3\ell',\mathbf{z}}(\tilde{W}^{\delta}_{3\ell'}(\mathbf{z}))^{*}\varphi\| \qquad \text{(F.14)}\end{aligned}$$

for any vector φ in the domain of the operator $\mathbf{p}+e\mathbf{A}$.

Let us estimate the first term in the right-hand side. Using the expression (5.9) of $W(\cdots)$, one has

$$\begin{aligned}
&\|\chi_{\ell'}(\mathbf{z})R_{5\ell',\mathbf{z}}\chi_{3\ell'}^{\delta}(\mathbf{z})(\tilde{W}_{3\ell'}^{\delta}(\mathbf{z}))^*\varphi\| \\
&\quad\le \|\chi_{\ell'}(\mathbf{z})R_{5\ell',\mathbf{z}}\chi_{3\ell'}^{\delta}(\mathbf{z})\upsilon_{3\ell'}(\mathbf{z})\varphi\| \\
&\qquad + \frac{\hbar}{m_e}\sum_{i=x,y}\|\chi_{\ell'}(\mathbf{z})R_{5\ell',\mathbf{z}}\phi_{i,3\ell'}(\mathbf{z})(p_i+eA_i)\varphi\|,
\end{aligned} \tag{F.15}$$

where we have written

$$\upsilon_{3\ell'}(\mathbf{z}) = \frac{\hbar^2}{2m_e}\Delta\chi_{3\ell'}^{\delta}(\mathbf{z}) + \delta V_{\omega,3\ell',3\ell'}\chi_{3\ell'}^{\delta}(\mathbf{z}). \tag{F.16}$$

Using the bound (F.1), the first term in the right-hand side can be estimated as

$$\begin{aligned}
\|\chi_{\ell'}(\mathbf{z})R_{5\ell',\mathbf{z}}\chi_{3\ell'}^{\delta}(\mathbf{z})\upsilon_{3\ell'}(\mathbf{z})\varphi\| &\le \sum_{\mathbf{u},\mathbf{v}}\|\chi_{\ell}(\mathbf{u})R_{5\ell',\mathbf{z}}\chi_{\ell}(\mathbf{v})\|\,\|\upsilon_{3\ell'}(\mathbf{z})\|_\infty\|\varphi\| \\
&\le 12\left(\frac{\ell'}{\ell}\right)^3(8e^{-\gamma\ell})^{\ell'/\ell-4}C_{\delta,\omega}\|R_{5\ell',\mathbf{z}}\|\,\|\varphi\|.
\end{aligned} \tag{F.17}$$

The summand in the right-hand side in (F.15) can be written as

$$\begin{aligned}
&\|\chi_{\ell'}(\mathbf{z})R_{5\ell',\mathbf{z}}\phi_{i,3\ell'}(\mathbf{z})(p_i+eA_i)\varphi\| \\
&\quad= \|\chi_{\ell'}(\mathbf{z})R_{5\ell',\mathbf{z}}W(\phi_{i,3\ell'}(\mathbf{z}))R_{5\ell',\mathbf{z}}(p_i+eA_i)\varphi\| \\
&\quad\le \frac{\hbar^2}{2m_e}\|\chi_{\ell'}(\mathbf{z})R_{5\ell',\mathbf{z}}(\Delta\phi_{i,3\ell'}(\mathbf{z}))R_{5\ell',\mathbf{z}}(p_i+eA_i)\varphi\| \\
&\qquad + \frac{\hbar}{m_e}\sum_{j=x,y}\|\chi_{\ell'}(\mathbf{z})R_{5\ell',\mathbf{z}}(\partial_j\phi_{i,3\ell'}(\mathbf{z}))(p_j+eA_j)R_{5\ell',\mathbf{z}}(p_i+eA_i)\varphi\| \\
&\quad\le 6\left(\frac{\ell'}{\ell}\right)^3(8e^{-\gamma\ell})^{\ell'/\ell-4}\|R_{5\ell',\mathbf{z}}\|\left[\frac{\hbar^2}{m_e}\max_{n=x,y}\{\|\Delta\phi_{n,3\ell'}(\mathbf{z})\|_\infty\}\|R_{5\ell',\mathbf{z}}(p_i+eA_i)\varphi\|.\right. \\
&\qquad \left. + \frac{4\hbar}{m_e}\max_{m,n=x,y}\{\|\partial_m\phi_{n,3\ell'}(\mathbf{z})\|_\infty\}\|(p_j+eA_j)R_{5\ell',\mathbf{z}}(p_i+eA_i)\varphi\|\right],
\end{aligned} \tag{F.18}$$

where we have used (5.9) and (F.1) again. From (5.19) and (5.20) with $\boldsymbol{\alpha} = (1,0)$ or $(0,1)$, one has

$$\|R(p_i+eA_i)\varphi\| \le \sqrt{2m_e}(1+f_{E,R})^{1/2}\|R\|^{1/2}\|\varphi\| \tag{F.19}$$

and

$$\|(p_j+eA_j)R(p_i+eA_i)\varphi\| \le 2m_e(1+f_{E,R})\|\varphi\|. \tag{F.20}$$

Combining these, (F.15), (F.17) and (F.18), we have

$$
\begin{aligned}
&\|\chi_{\ell'}(\mathbf{z})R_{5\ell',\mathbf{z}}\chi^{\delta}_{3\ell'}(\mathbf{z})(\tilde{W}^{\delta}_{3\ell'}(\mathbf{z}))^*\varphi\| \\
&\quad\le 6\left(\frac{\ell'}{\ell}\right)^3(8e^{-\gamma\ell})^{\ell'/\ell-4}f_4(|E|,\|R_{5\ell',\mathbf{z}}\|)\|R_{5\ell',\mathbf{z}}\|\|\varphi\|.
\end{aligned}
\tag{F.21}
$$

Next let us estimate the second term in the right-hand side of (F.14). Note that

$$
\begin{aligned}
&\|(\tilde{W}^{\delta}_{3\ell'}(\mathbf{z}))^*R_{3\ell',\mathbf{z}}(\tilde{W}^{\delta}_{3\ell'}(\mathbf{z}))^*\varphi\| \\
&\le C^2_{\delta,\omega}\|R_{3\ell',\mathbf{z}}\|\|\varphi\| + \frac{\hbar}{m_e}C_{\delta,\omega}\|R_{3\ell',\mathbf{z}}(\mathbf{p}+e\mathbf{A})\cdot(\nabla\chi^{\delta}_{3\ell'}(\mathbf{z}))\varphi\| \\
&\quad + \frac{\hbar}{m_e}C_{\delta,\omega}\max_{m=x,y}\{\|\partial_m\chi^{\delta}_{3\ell'}(\mathbf{z})\|_\infty\}\sum_{i=x,y}\|(p_i+eA_i)R_{3\ell',\mathbf{z}}\|\|\varphi\| \\
&\quad + \frac{\hbar^2}{m_e^2}\max_{m=x,y}\{\|\partial_m\chi^{\delta}_{3\ell'}(\mathbf{z})\|_\infty\} \\
&\quad \times\sum_{i=x,y}\|(p_i+eA_i)R_{3\ell',\mathbf{z}}(\mathbf{p}+e\mathbf{A})\cdot(\nabla\chi^{\delta}_{3\ell'}(\mathbf{z}))\varphi\|,
\end{aligned}
\tag{F.22}
$$

where we have used (5.9). Thus we have

$$
\begin{aligned}
&\|\chi_{\ell'}(\mathbf{z})R_{5\ell',\mathbf{z}}(\tilde{W}^{\delta}_{3\ell'}(\mathbf{z}))^*R_{3\ell',\mathbf{z}}(\tilde{W}^{\delta}_{3\ell'}(\mathbf{z}))^*\varphi\| \\
&\quad\le 6\left(\frac{\ell'}{\ell}\right)^3(8e^{-\gamma\ell})^{\ell'/\ell-4}f_5(|E|,\|R_{3\ell',\mathbf{z}}\|)\|R_{5\ell',\mathbf{z}}\|\|\varphi\|
\end{aligned}
\tag{F.23}
$$

in the same way. Substituting this and (F.21) into (F.14), the desired bound (F.9) is obtained. □

Similarly, one has the following lemma:

Lemma F.3 *Let ℓ,ℓ' be odd integers larger than 1 such that ℓ' is a multiple of ℓ and satisfies $\ell'>4\ell$. Assume that the event A^{good} given in Lemma F.1 occurs. Then*

$$
\begin{aligned}
&\|\tilde{W}^{\delta}_{3\ell'}(\mathbf{z})R_{3\ell',\mathbf{z}}\chi_{\ell'}(\mathbf{z})\| \\
&\quad\le 6\left(\frac{\ell'}{\ell}\right)^3(8e^{-\gamma\ell})^{\ell'/\ell-4}[f_6(|E|,\|R_{5\ell',\mathbf{z}}\|)+f_5(|E|,\|R_{3\ell',\mathbf{z}}\|)]\|R_{5\ell',\mathbf{z}}\|,
\end{aligned}
\tag{F.24}
$$

where the function f_6 is given by

$$
\begin{aligned}
f_6(|E|,\|R\|) &= 2C_{\delta,\omega}+\frac{8\hbar^2}{m_e}\max_{m=x,y}\{\||\nabla\phi_{m,3\ell'}(\mathbf{z})|\|_\infty\}(1+f_{E,R}) \\
&\quad + 2\sqrt{2}\left(\frac{\hbar}{\sqrt{m_e}}\right)^3\max_{n=x,y}\{\|\Delta\phi_{n,3\ell'}(\mathbf{z})\|_\infty\}(1+f_{E,R})^{1/2}\|R\|^{1/2}.
\end{aligned}
\tag{F.25}
$$

Proof Using the geometric resolvent equation, one has

$$
\begin{aligned}
&\tilde{W}^{\delta}_{3\ell'}(\mathbf{z})\chi^{\delta}_{3\ell'}(\mathbf{z})R_{5\ell',\mathbf{z}}\chi_{\ell'}(\mathbf{z}) \\
&\quad=\tilde{W}^{\delta}_{3\ell'}(\mathbf{z})R_{3\ell',\mathbf{z}}\chi_{\ell'}(\mathbf{z})+\tilde{W}^{\delta}_{3\ell'}(\mathbf{z})R_{3\ell',\mathbf{z}}\tilde{W}^{\delta}_{3\ell'}(\mathbf{z})R_{5\ell',\mathbf{z}}\chi_{\ell'}(\mathbf{z}).
\end{aligned}
\tag{F.26}
$$

Therefore

$$\|\tilde{W}^{\delta}_{3\ell'}(\mathbf{z})R_{3\ell',\mathbf{z}}\chi_{\ell'}(\mathbf{z})\| \le \|\tilde{W}^{\delta}_{3\ell'}(\mathbf{z})\chi^{\delta}_{3\ell'}(\mathbf{z})R_{5\ell',\mathbf{z}}\chi_{\ell'}(\mathbf{z})\| \\ + \|\tilde{W}^{\delta}_{3\ell'}(\mathbf{z})R_{3\ell',\mathbf{z}}\tilde{W}^{\delta}_{3\ell'}(\mathbf{z})R_{5\ell',\mathbf{z}}\chi_{\ell'}(\mathbf{z})\|. \tag{F.27}$$

Since the second term in the right-hand side can be estimated in the same way as in the proof of the preceding lemma, it is enough to estimate the first term. Using (5.9), one has

$$\begin{aligned} &\|\tilde{W}^{\delta}_{3\ell'}(\mathbf{z})\chi^{\delta}_{3\ell'}(\mathbf{z})R_{5\ell',\mathbf{z}}\chi_{\ell'}(\mathbf{z})\| \\ &\quad \le \|\upsilon_{3\ell'}(\mathbf{z})\chi^{\delta}_{3\ell'}(\mathbf{z})R_{5\ell',\mathbf{z}}\chi_{\ell'}(\mathbf{z})\| \\ &\qquad + \frac{\hbar}{m_e}\sum_{i=x,y}\|(p_i+eA_i)(\partial_i\chi^{\delta}_{3\ell'}(\mathbf{z}))\chi^{\delta}_{3\ell'}(\mathbf{z})R_{5\ell',\mathbf{z}}\chi_{\ell'}(\mathbf{z})\|. \end{aligned} \tag{F.28}$$

The first term in the right-hand side can be estimated by using the bound (F.1). The operators in the sum in the right-hand side can be written as

$$\begin{aligned} &(p_i+eA_i)\phi_{i,3\ell'}(\mathbf{z})R_{5\ell',\mathbf{z}}\chi_{\ell'}(\mathbf{z}) \\ &\quad = (p_i+eA_i)R_{5\ell',\mathbf{z}}W(\phi_{i,3\ell'}(\mathbf{z}))R_{5\ell',\mathbf{z}}\chi_{\ell'}(\mathbf{z}) \\ &\quad = \frac{\hbar^2}{2m_e}(p_i+eA_i)R_{5\ell',\mathbf{z}}(\Delta\phi_{i,3\ell'}(\mathbf{z}))R_{5\ell',\mathbf{z}}\chi_{\ell'}(\mathbf{z}) \\ &\qquad - \frac{i\hbar}{m_e}(p_i+eA_i)R_{5\ell',\mathbf{z}}(\mathbf{p}+e\mathbf{A})\cdot(\nabla\phi_{i,3\ell'}(\mathbf{z}))R_{5\ell',\mathbf{z}}\chi_{\ell'}(\mathbf{z}), \end{aligned} \tag{F.29}$$

where we have used (5.9) again. Similarly the norm of the operators in this right-hand side can be estimated by using (5.19), (E.1) and (F.1). □

Proof of Lemma 6.2 Assume that the event A^{good} given in Lemma F.1 occurs. Then, from the preceding three lemmas, one has

$$\|\chi_{\ell'}R_{3\ell,\mathbf{z}}(\tilde{W}^{\delta}_{3\ell'}(\mathbf{z}))^*\| \le \text{Const}\times(\ell')^3\exp[-\ell'\{\gamma(1-4\ell/\ell')-3\log 2/\ell\}] \\ \times|E|(\|R_{3\ell,\mathbf{z}}\|+\|R_{5\ell,\mathbf{z}}\|)\|R_{5\ell,\mathbf{z}}\| \tag{F.30}$$

and

$$\|\tilde{W}^{\delta}_{3\ell'}(\mathbf{z})R_{3\ell,\mathbf{z}}\chi_{\ell'}\| \le \text{Const}\times(\ell')^3\exp[-\ell'\{\gamma(1-4\ell/\ell')-3\log 2/\ell\}] \\ \times|E|(\|R_{3\ell,\mathbf{z}}\|+\|R_{5\ell,\mathbf{z}}\|)\|R_{5\ell,\mathbf{z}}\| \tag{F.31}$$

with the probability larger than $1-(5\ell'/\ell)^4\eta^2$ for a large $|E|$ and for large $\|R_{3\ell,\mathbf{z}}\|$, $\|R_{5\ell,\mathbf{z}}\|$.

In the Wegner estimate (A.38), we choose

$$(\delta E)^{-1} = C_{\mathrm{W}}K_3\|g\|_\infty|\Lambda_{5\ell'}|\times 4(\ell')^{\xi}. \tag{F.32}$$

Then, for $q=3,5$, one has

$$\|R_{q\ell,\mathbf{z}}\| \le (\delta E)^{-1} \tag{F.33}$$

with the probability larger than $1-(\ell')^{-\xi}/4$. Clearly, one has

$$\|R_{q\ell,\mathbf{z}}\| \le \text{Const} \times K_3(\ell')^{\xi+2} \tag{F.34}$$

for a large ℓ'. Since the probability that each event occurs is larger than $1-(\ell')^{-\xi}/4$, the probability that the two events simultaneously occur is larger than $1-(\ell')^{-\xi}/2$.

From these observations, the right-hand side of (F.31) can be bounded from above by

$$\exp[-\ell'\{\gamma(1-4\ell/\ell')-3\log 2/\ell-\log(\text{Const}\times K_3^2|E|)/\ell'-(2\xi+7)\log\ell'/\ell'\}] \tag{F.35}$$

with the probability larger than $1-(5\ell'/\ell)^4\eta^2-(\ell')^{-\xi}/2$ for a large $|E|$ and for a large ℓ'. Since one has

$$\frac{3\log 2}{\ell}+\frac{\log(\text{Const}\times K_3^2|E|)}{\ell'} \le \frac{\log(c_0K_3^2|E|)}{\ell} \tag{F.36}$$

with a positive constant c_0, the proof of the lemma is completed. □

Proof of Lemma 6.3 Take $\ell'=\ell_{k+1}$ and $\ell=\ell_k$ in Lemma 6.2, and assume that $\Lambda_{3\ell_k}(\cdots)$ is a γ_k-good box with the probability larger than $1-\eta$ with $\eta=(\ell_k)^{-\xi}$. From the definition (6.8) of ℓ_{k+1}, we have

$$\frac{\ell_{k+1}}{\ell_k}=[\ell_k^{1/2}]_{\text{odd}}^{\ge}=\ell_k^{1/2}+\delta\ell_k \quad \text{with } 0\le\delta\ell_k<2. \tag{F.37}$$

Using this identity, η' of (6.6) can be written as

$$\begin{aligned}
\eta'=\eta_{k+1} &= 5^4\left(\frac{\ell_{k+1}}{\ell_k}\right)^4(\ell_k)^{-2\xi}+\frac{1}{2}(\ell_{k+1})^{-\xi} \\
&= (\ell_{k+1})^{-\xi}\left[5^4\left(\frac{\ell_{k+1}}{\ell_k}\right)^{4+\xi}(\ell_k)^{-\xi}+\frac{1}{2}\right] \\
&= (\ell_{k+1})^{-\xi}\left[5^4(\ell_k)^{2-\xi/2}\left(1+\frac{\delta\ell_k}{\ell_k^{1/2}}\right)^{4+\xi}+\frac{1}{2}\right].
\end{aligned} \tag{F.38}$$

Therefore, if the initial scale ℓ_0 satisfies

$$5^4(\ell_0)^{2-\xi/2}(1+2\ell_0^{-1/2})^{4+\xi}\le 1/2, \tag{F.39}$$

then we have $\eta_{k+1}\le(\ell_{k+1})^{-\xi}$. Actually this inequality holds for a large ℓ_0 because of the assumption, $\xi>4$.

Next we define γ_k inductively according to (6.7) as

$$\gamma_{k+1}=\gamma_k(1-4\ell_k/\ell_{k+1})-d_k \tag{F.40}$$

with

$$d_k=\frac{\log(c_0K_3^2|E|)}{\ell_k}+\frac{(2s+7)\log\ell_{k+1}}{\ell_{k+1}}. \tag{F.41}$$

One can easily find

$$\begin{aligned}\gamma_{k+1} &= \left(1-\frac{4\ell_k}{\ell_{k+1}}\right)\left(1-\frac{4\ell_{k-1}}{\ell_k}\right)\cdots\left(1-\frac{4\ell_1}{\ell_2}\right)\left(1-\frac{4\ell_0}{\ell_1}\right)\gamma_0\\&\quad - d_k - \left(1-\frac{4\ell_k}{\ell_{k+1}}\right)d_{k-1} - \left(1-\frac{4\ell_k}{\ell_{k+1}}\right)\left(1-\frac{4\ell_{k-1}}{\ell_k}\right)d_{k-2}\cdots\\&\quad - \left(1-\frac{4\ell_k}{\ell_{k+1}}\right)\left(1-\frac{4\ell_{k-1}}{\ell_k}\right)\cdots\left(1-\frac{4\ell_1}{\ell_2}\right)d_0\\&\ge \gamma_0\prod_{j=0}^{k}\left(1-\frac{4\ell_j}{\ell_{j+1}}\right) - \sum_{j=0}^{k} d_j.\end{aligned} \tag{F.42}$$

Note that

$$\ell_k \ge (\ell_{k-1})^{3/2} \ge (\ell_{k-2})^{(3/2)^2} \ge \cdots \ge (\ell_0)^{(3/2)^k}, \tag{F.43}$$

and one has

$$\frac{\ell_k}{\ell_{k+1}} \le \frac{1}{\ell_k^{1/2}} \le \exp\left[-\frac{1}{2}(3/2)^k \log \ell_0\right]. \tag{F.44}$$

Using this inequality, the product in the right-hand side of (F.42) can be evaluated as

$$\prod_{j=0}^{k}\left(1-\frac{4\ell_j}{\ell_{j+1}}\right) \ge \prod_{j=0}^{k}\left\{1-4\exp\left[-\frac{1}{2}(3/2)^j \log \ell_0\right]\right\}. \tag{F.45}$$

This right-hand side is uniformly bounded from below by some positive constant if $\ell_0 > 16$. The sum of d_k in the right-hand side of (F.42) can also be evaluated as

$$\begin{aligned}\sum_{j=0}^{k} d_j &\le \log(c_0 K_3^2 |E|)\sum_{j=0}^{k}\exp[-(3/2)^j \log \ell_0]\\&\quad + (2s+7)\log \ell_0 \sum_{j=0}^{k}(3/2)^{j+1}\exp[-(3/2)^{j+1}\log \ell_0].\end{aligned} \tag{F.46}$$

This right-hand side becomes small for B large enough because of $K_3 = \mathcal{O}(B)$ or $\mathcal{O}(1)$, $|E| = \mathcal{O}(B)$ and $\ell_0 = \mathcal{O}(B^{1/2})$. As a result, γ_k is uniformly bounded from below by some positive constant γ_∞. □

Appendix G: Proof of Lemma 9.2

The difference between (9.18) and (9.19) is estimated by

$$\Delta I := \frac{1}{\mathcal{V}_\ell}\left(\sum_{\mathbf{a}\in\Lambda_\ell}\sum_{\mathbf{u}\in(\mathbf{Z}\varepsilon^2)^*\setminus\Lambda_\ell^*}\sum_{\mathbf{v},\mathbf{w}} + \sum_{\mathbf{a}\in\mathbf{Z}_\varepsilon^2\setminus\Lambda_\ell}\sum_{\mathbf{u}\in\Lambda_\ell^*}\sum_{\mathbf{v},\mathbf{w}} |t_{\mathbf{u},\mathbf{v}}t_{\mathbf{v},\mathbf{w}}t_{\mathbf{w},\mathbf{u}}||S_{\mathbf{u},\mathbf{v},\mathbf{w},\mathbf{u}}|\right). \tag{G.1}$$

In the same way as in the proof of Theorem 8.5, it is sufficient to show $\mathbf{E}[\Delta I] \to 0$ as $\ell \to \infty$.

To begin with, we note that

$$\begin{aligned}
&|\,\mathrm{Tr}\,\chi_\varepsilon(\mathbf{u})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{v})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{w})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{u})| \\
&\quad\le \sqrt{\mathrm{Tr}\,\chi_\varepsilon(\mathbf{u})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{v})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{v})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{u})}\sqrt{\mathrm{Tr}\,\chi_\varepsilon(\mathbf{u})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{w})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{w})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{u})} \\
&\quad\le \|\chi_\varepsilon(\mathbf{u})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{v})\|\,\|\chi_\varepsilon(\mathbf{u})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{w})\|\sqrt{\mathrm{Tr}\,\chi_\varepsilon(\mathbf{v})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{v})}\sqrt{\mathrm{Tr}\,\chi_\varepsilon(\mathbf{w})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{w})} \\
&\quad\le \mathrm{Const}\times\|\chi_\varepsilon(\mathbf{u})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{v})\|\,\|\chi_\varepsilon(\mathbf{u})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{w})\|,
\end{aligned} \tag{G.2}$$

where we have used (8.27). Further, Schwarz's inequality yields

$$\begin{aligned}
&\mathbf{E}[\|\chi_\varepsilon(\mathbf{u})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{v})\|\,\|\chi_\varepsilon(\mathbf{u})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{w})\|] \\
&\quad\le \mathbf{E}[\|\chi_\varepsilon(\mathbf{u})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{v})\|^2]^{1/2}\mathbf{E}[\|\chi_\varepsilon(\mathbf{u})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{w})\|^2]^{1/2} \\
&\quad\le \mathbf{E}[\|\chi_\varepsilon(\mathbf{u})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{v})\|]^{1/2}\mathbf{E}[\|\chi_\varepsilon(\mathbf{u})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{w})\|]^{1/2},
\end{aligned} \tag{G.3}$$

where we have used $\|\chi_\varepsilon(\mathbf{u})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{v})\|\le 1$ for any $\mathbf{u},\mathbf{v}$. From these observations and the decay bound (7.17) for the Fermi sea projection, it is sufficient to estimate

$$\frac{1}{\mathcal{V}_\ell}\Bigg(\sum_{\mathbf{a}\in\Lambda_\ell}\sum_{\mathbf{u}\in(\mathbf{Z}_\varepsilon^2)^*\setminus\Lambda_\ell^*}\sum_{\mathbf{v},\mathbf{w}}+\sum_{\mathbf{a}\in\mathbf{Z}_\varepsilon^2\setminus\Lambda_\ell}\sum_{\mathbf{u}\in\Lambda_\ell^*}\sum_{\mathbf{v},\mathbf{w}}|t_{\mathbf{u},\mathbf{v}}t_{\mathbf{v},\mathbf{w}}t_{\mathbf{w},\mathbf{u}}|e^{-\mu|\mathbf{u}-\mathbf{v}|/2}e^{-\mu|\mathbf{u}-\mathbf{w}|/2}\Bigg). \tag{G.4}$$

Consider first the case with $|\mathbf{u}-\mathbf{a}|\le\varepsilon_1\ell^\delta$ with $\delta\in(0,1)$. Using the bound,

$$|t_{\mathbf{u},\mathbf{v}}t_{\mathbf{v},\mathbf{w}}t_{\mathbf{w},\mathbf{u}}|\le 2^3\frac{|\mathbf{u}-\mathbf{v}||\mathbf{u}-\mathbf{w}|}{|\mathbf{u}-\mathbf{a}|^2}, \tag{G.5}$$

which is derived from (9.5), we have

$$\sum_{\mathbf{v},\mathbf{w}}|t_{\mathbf{u},\mathbf{v}}t_{\mathbf{v},\mathbf{w}}t_{\mathbf{w},\mathbf{u}}|e^{-\mu|\mathbf{u}-\mathbf{v}|/2}e^{-\mu|\mathbf{v}-\mathbf{w}|/2}<\mathrm{Const}\times\frac{1}{|\mathbf{u}-\mathbf{a}|^2}. \tag{G.6}$$

Therefore the corresponding error is estimated by

$$\begin{aligned}
&\frac{1}{\ell^2}\Bigg(\sum_{\substack{\mathbf{a}\in\Lambda_\ell,\mathbf{u}\in(\mathbf{Z}_\varepsilon^2)^*\setminus\Lambda_\ell^*:\\ |\mathbf{u}-\mathbf{a}|\le\varepsilon_1\ell^\delta}}+\sum_{\substack{\mathbf{a}\in\mathbf{Z}_\varepsilon^2\setminus\Lambda_\ell,\mathbf{u}\in\Lambda_\ell^*:\\ |\mathbf{u}-\mathbf{a}|\le\varepsilon_1\ell^\delta}}\frac{1}{|\mathbf{u}-\mathbf{a}|^2}\Bigg) \\
&\quad\le\frac{\mathrm{Const}\times\ell\cdot\ell^\delta(\mathrm{Const}+\mathrm{Const}\times\log\ell)}{\ell^2}.
\end{aligned} \tag{G.7}$$

This vanishes as $\ell\to\infty$.

When $|\mathbf{u}-\mathbf{a}|>\varepsilon_1\ell^\delta$, we further decompose it into two cases: (i) both $\mathbf{v}$ and $\mathbf{w}$ fall into inside the ball with radius $|\mathbf{u}-\mathbf{a}|$ around $\mathbf{u}$, i.e., $|\mathbf{v}-\mathbf{u}|<|\mathbf{u}-\mathbf{a}|$ and $|\mathbf{w}-\mathbf{u}|<|\mathbf{u}-\mathbf{a}|$, (ii) one of $\mathbf{v}$ or $\mathbf{w}$ falls into outside the ball, i.e., $|\mathbf{v}-\mathbf{u}|\ge|\mathbf{u}-\mathbf{a}|$ or $|\mathbf{w}-\mathbf{u}|\ge|\mathbf{u}-\mathbf{a}|$. The latter contribution is exponentially small in ℓ^δ. Actually, one has

$$\begin{aligned}
&\sum_{\mathbf{v},\mathbf{w}\,\mathrm{satisfy\,(ii)}}|t_{\mathbf{u},\mathbf{v}}t_{\mathbf{v},\mathbf{w}}t_{\mathbf{w},\mathbf{u}}|e^{-\mu|\mathbf{u}-\mathbf{v}|/2}e^{-\mu|\mathbf{u}-\mathbf{w}|/2} \\
&\quad\le 2^3\Bigg[\sum_{\substack{\mathbf{v}:|\mathbf{v}-\mathbf{u}|\ge|\mathbf{u}-\mathbf{a}|,\\ \mathbf{w}}}+\sum_{\substack{\mathbf{w}:|\mathbf{w}-\mathbf{u}|\ge|\mathbf{u}-\mathbf{a}|,\\ \mathbf{v}}}e^{-\mu|\mathbf{u}-\mathbf{v}|/2}e^{-\mu|\mathbf{u}-\mathbf{w}|/2}\Bigg] \\
&\quad\le\mathrm{Const}\times e^{-\mu'|\mathbf{u}-\mathbf{a}|}
\end{aligned} \tag{G.8}$$

with a positive constant μ'.

Finally, consider the former case (i). To begin with, we note that

$$t_{\mathbf{u},\mathbf{v}}t_{\mathbf{v},\mathbf{w}}t_{\mathbf{w},\mathbf{u}} = 2i\{\sin\angle(\mathbf{u},\mathbf{a},\mathbf{v}) + \sin\angle(\mathbf{v},\mathbf{a},\mathbf{w}) + \sin\angle(\mathbf{w},\mathbf{a},\mathbf{u})\}. \tag{G.9}$$

We write $\alpha = \angle(\mathbf{u},\mathbf{a},\mathbf{v})$, $\beta = \angle(\mathbf{v},\mathbf{a},\mathbf{w})$ and $\gamma = \angle(\mathbf{w},\mathbf{a},\mathbf{u})$ for short. In this case, one notices that $\alpha, \beta \in (-\pi/2, \pi/2)$ and $\alpha + \beta + \gamma = 0$. From these, one has

$$\begin{aligned} |\sin\alpha + \sin\beta + \sin\gamma| &\le 2\left(|\sin\alpha|\sin^2\frac{\beta}{2} + |\sin\beta|\sin^2\frac{\alpha}{2}\right) \\ &\le 2(|\sin\alpha|\sin^2\beta + |\sin\beta|\sin^2\alpha) \\ &\le \frac{2}{|\mathbf{u}-\mathbf{a}|^3}(|\mathbf{u}-\mathbf{v}||\mathbf{w}-\mathbf{u}|^2 + |\mathbf{v}-\mathbf{u}|^2|\mathbf{w}-\mathbf{u}|), \end{aligned} \tag{G.10}$$

where we have used

$$|\sin\alpha| \le \frac{|\mathbf{v}-\mathbf{u}|}{|\mathbf{u}-\mathbf{a}|} \quad \text{and} \quad |\sin\beta| \le \frac{|\mathbf{w}-\mathbf{u}|}{|\mathbf{u}-\mathbf{a}|} \tag{G.11}$$

for getting the third inequality. From these observations, we obtain

$$\sum_{\mathbf{v},\mathbf{w}} |t_{\mathbf{u},\mathbf{v}}t_{\mathbf{v},\mathbf{w}}t_{\mathbf{w},\mathbf{u}}| e^{-\mu|\mathbf{u}-\mathbf{v}|/2} e^{-\mu|\mathbf{u}-\mathbf{w}|/2} \le \text{Const} \times \frac{1}{|\mathbf{u}-\mathbf{a}|^3}. \tag{G.12}$$

The corresponding contribution is estimated by

$$\frac{1}{\ell^2}\left(\sum_{\substack{\mathbf{a}\in\Lambda_\ell, \mathbf{u}\in(\mathbf{Z}_\varepsilon^2)^*\setminus\Lambda_\ell^*: \\ |\mathbf{u}-\mathbf{a}|>\varepsilon_1\ell^\delta}} + \sum_{\substack{\mathbf{a}\in\mathbf{Z}_\varepsilon^2\setminus\Lambda_\ell, \mathbf{u}\in\Lambda_\ell^*: \\ |\mathbf{u}-\mathbf{a}|>\varepsilon_1\ell^\delta}} \frac{1}{|\mathbf{u}-\mathbf{a}|^3}\right) \le \frac{\text{Const}}{\ell^\delta}. \tag{G.13}$$

This vanishes as $\ell \to \infty$.

Appendix H: Proof of Lemma 9.7

In order to prove Lemma 9.7, we introduce a partition $\{\chi_b^\delta(\mathbf{u})\}_\mathbf{u}$ of unity and prepare Lemma H.1 below. Here $\chi_b^\delta(\mathbf{u})$ are C^2 positive functions with a compact support such that $\sum_\mathbf{u} \chi_b^\delta(\mathbf{u}) = 1$. Let $\tilde{\chi}_b(\mathbf{u})$ denote the characteristic function of the support of $\chi_b^\delta(\mathbf{u})$.

Lemma H.1 *The following bound is valid*:

$$\begin{aligned} \|(p_s + eA_s)\chi_b^\delta(\mathbf{u})R(z)\chi_\varepsilon(\mathbf{v})\| &\le [\text{Const} + (2m_e|z|)^{1/2}]\|\tilde{\chi}_b(\mathbf{u})R(z)\chi_\varepsilon(\mathbf{v})\| \\ &\quad + \sqrt{2m_e}\|\chi_b^\delta(\mathbf{u})R(z)\chi_\varepsilon(\mathbf{v})\|^{1/2}, \end{aligned} \tag{H.1}$$

where the positive constant depends only on the strengths of the potentials, V_0, V_ω *and on the cutoff functions* $\chi_b^\delta(\mathbf{u})$.

Proof Let φ be a vector on $\mathbf{R}^2$. Then one has

$$\begin{aligned}
&(\varphi, \chi_\varepsilon(\mathbf{v}) R^*(z) \chi_b^\delta(\mathbf{u})(p_s + eA_s)^2 \chi_b^\delta(\mathbf{u}) R(z) \chi_\varepsilon(\mathbf{v})\varphi) \\
&\quad \le 2m_e(\|V_0\|_\infty + \|V_\omega\|_\infty)\|\chi_b^\delta(\mathbf{u}) R(z)\chi_\varepsilon(\mathbf{v})\varphi\|^2 \\
&\qquad + 2m_e(\varphi, \chi_\varepsilon(\mathbf{v}) R^*(z)\chi_b^\delta(\mathbf{u}) H_\omega \chi_b^\delta(\mathbf{u}) R(z)\chi_\varepsilon(\mathbf{v})\varphi)
\end{aligned} \tag{H.2}$$

by using the inequality $(p_s + eA_s)^2/(2m_e) \le H_\omega + \|V_0\|_\infty + \|V_\omega\|_\infty$. Further, the second term in the right-hand side is evaluated as

$$\begin{aligned}
&2m_e(\varphi, \chi_\varepsilon(\mathbf{v}) R^*(z)\chi_b^\delta(\mathbf{u}) H_\omega \chi_b^\delta(\mathbf{u}) R(z)\chi_\varepsilon(\mathbf{v})\varphi) \\
&\quad \le 2m_e|z|\|\chi_b^\delta(\mathbf{u}) R(z)\chi_\varepsilon(\mathbf{v})\varphi\|^2 \\
&\qquad + 2\hbar \sum_{s=x,y} \|[\partial_s \chi_b^\delta(\mathbf{u})] R(z)\chi_\varepsilon(\mathbf{v})\varphi\| \|\chi_b^\delta(\mathbf{u})(p_s + eA_s) R(z)\chi_\varepsilon(\mathbf{v})\varphi\| \\
&\qquad + \hbar^2 \|\chi_b^\delta(\mathbf{u}) R(z)\chi_\varepsilon(\mathbf{v})\varphi\| \|[\Delta \chi_b^\delta(\mathbf{u})] R(z)\chi_\varepsilon(\mathbf{v})\varphi\| \\
&\qquad + 2m_e\|\chi_b^\delta(\mathbf{u}) R(z)\chi_\varepsilon(\mathbf{v})\varphi\| \|\chi_b^\delta(\mathbf{u})\chi_\varepsilon(\mathbf{v})\varphi\|
\end{aligned} \tag{H.3}$$

by using

$$H_\omega \chi_b^\delta(\mathbf{u}) = -\frac{i\hbar}{m_e}\nabla\chi_b^\delta(\mathbf{u}) \cdot (\mathbf{p} + e\mathbf{A}) - \frac{\hbar^2}{2m_e}\Delta\chi_b^\delta(\mathbf{u}) + \chi_b^\delta(\mathbf{u}) H_\omega. \tag{H.4}$$

Combing these two bounds, we obtain

$$\begin{aligned}
&\|(p_s + eA_s)\chi_b^\delta(\mathbf{u}) R(z)\chi_\varepsilon(\mathbf{v})\|^2 \\
&\quad \le \left[2m_e(\|V_0\|_\infty + \|V_\omega\|_\infty + |z|) + 2\hbar^2 \sum_{s=x,y} \|\partial_s \chi_b^\delta(\mathbf{u})\|_\infty^2 + \hbar^2 \|\Delta\chi_b^\delta(\mathbf{u})\|_\infty\right] \\
&\qquad \times \|\tilde{\chi}_b(\mathbf{u}) R(z)\chi_\varepsilon(\mathbf{v})\|^2 \\
&\qquad + 2\hbar \sum_{s=x,y} \|\partial_s \chi_b^\delta(\mathbf{u})\|_\infty \|\tilde{\chi}_b(\mathbf{u}) R(z)\chi_\varepsilon(\mathbf{v})\| \|(p_s + eA_s)\chi_b^\delta(\mathbf{u}) R(z)\chi_\varepsilon(\mathbf{v})\| \\
&\qquad + 2m_e\|\chi_b^\delta(\mathbf{u}) R(z)\chi_\varepsilon(\mathbf{v})\varphi\| \|\chi_b^\delta(\mathbf{u})\chi_\varepsilon(\mathbf{v})\|.
\end{aligned} \tag{H.5}$$

Solving this quadratic inequality and using the inequality $\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$ for $a, b \ge 0$, the desired bound is obtained. □

Proof of Lemma 9.7 Note that

$$\begin{aligned}
\mathbf{E}[|\mathcal{I}(P_{\mathrm{F}}; \Omega, \ell_{\mathrm{P}}) - \mathcal{I}(P_{\mathrm{F},\Lambda}; \Omega)|] &= \mathbf{E}[|\mathcal{I}(P_{\mathrm{F}}; \Omega, \ell_{\mathrm{P}}) - \mathcal{I}(P_{\mathrm{F},\Lambda}; \Omega, \ell_{\mathrm{P}})|\mathbf{I}(M_\Lambda)] \\
&\quad + \mathbf{E}[|\mathcal{I}(P_{\mathrm{F}}; \Omega, \ell_{\mathrm{P}}) - \mathcal{I}(P_{\mathrm{F},\Lambda}; \Omega, \ell_{\mathrm{P}})|\mathbf{I}(M_\Lambda^c)],
\end{aligned} \tag{H.6}$$

where M_Λ is the event which was introduced in the proof of Lemma 8.1, and $\mathbf{I}(A)$ is the indicator function of an event A. The second term in the right-hand side is vanishing in the limit $L \uparrow \infty$ as

$$\begin{aligned}
&\mathbf{E}[|\mathcal{I}(P_{\mathrm{F}};\Omega,\ell_{\mathrm{P}})-\mathcal{I}(P_{\mathrm{F},\Lambda};\Omega)|\mathbf{I}(M_\Lambda^c)]\\
&\quad\le \mathbf{E}[|\mathcal{I}(P_{\mathrm{F}};\Omega,\ell_{\mathrm{P}})-\mathcal{I}(P_{\mathrm{F},\Lambda};\Omega)|^2]^{1/2}\mathrm{E}[\mathbf{I}(M_\Lambda^c)]^{1/2}\\
&\quad\le \mathrm{Const}\times L^{-2[\kappa(\xi+2)-3]/6},
\end{aligned}\tag{H.7}$$

where we have used Schwarz's inequality for getting the first inequality, and we have used[12] Lemma 8.4 and $\mathrm{Prob}(M_\Lambda^c)\le \mathrm{Const}\times L^{-2[\kappa(\xi+2)-3]/3}$ for the second inequality.

In order to estimate the first term in the right-hand side of (H.6), we first note that

$$\mathrm{Tr}\,\chi_\Omega P_{\mathrm{F}}[[P_{\mathrm{F}},x],[P_{\mathrm{F}},y]]\chi_\Omega=\mathrm{Tr}\,\chi_\Omega(P_{\mathrm{F}}xP_{\mathrm{F}}yP_{\mathrm{F}}-P_{\mathrm{F}}yP_{\mathrm{F}}xP_{\mathrm{F}})\chi_\Omega.\tag{H.8}$$

We want to rewrite this right-hand side. We have

$$\begin{aligned}
\mathrm{Tr}\,\chi_\Omega P_{\mathrm{F}}xP_{\mathrm{F}}yP_{\mathrm{F}}\chi_\Omega&=\mathrm{Tr}\,\chi_\Omega P_{\mathrm{F}}x(\chi_\Lambda^\delta+1-\chi_\Lambda^\delta)P_{\mathrm{F}}y(\chi_\Lambda^\delta+1-\chi_\Lambda^\delta)P_{\mathrm{F}}\chi_\Omega\\
&=\mathrm{Tr}\,\chi_\Omega P_{\mathrm{F}}x\chi_\Lambda^\delta P_{\mathrm{F}}y\chi_\Lambda^\delta P_{\mathrm{F}}\chi_\Omega+\text{corrections}.
\end{aligned}\tag{H.9}$$

The contributions from the corrections decay exponentially in the distance between Ω and the support of $1-\chi_\Lambda^\delta$ by the bound (7.17) for the Fermi sea projection P_{F}. The rest is written

$$\mathrm{Tr}\,\chi_\Omega P_{\mathrm{F}}x\chi_\Lambda^\delta P_{\mathrm{F}}y\chi_\Lambda^\delta P_{\mathrm{F}}\chi_\Omega=\frac{1}{2\pi i}\int dz\,\mathrm{Tr}\,\chi_\Omega R(z)x\chi_\Lambda^\delta P_{\mathrm{F}}y\chi_\Lambda^\delta P_{\mathrm{F}}\chi_\Omega.\tag{H.10}$$

Note that one has

$$\chi_\Omega R(z)=\chi_\Omega\chi_\Lambda^\delta R(z)=\chi_\Omega R_\Lambda(z)\chi_\Lambda^\delta+\chi_\Omega R_\Lambda(z)\tilde{W}(\chi_\Lambda^\delta)R(z)\tag{H.11}$$

from the geometric resolvent equation, $\chi_\Lambda^\delta R(z)=R_\Lambda(z)\chi_\Lambda^\delta+R_\Lambda(z)\tilde{W}(\chi_\Lambda^\delta)R(z)$, where $\tilde{W}(\chi_\Lambda^\delta)=W(\chi_\Lambda^\delta)+(\tilde{V}_{\omega,\Lambda}-V_\omega)\chi_\Lambda^\delta$, and $\tilde{V}_{\omega,\Lambda}$ is the slightly modified potential near the boundary of Λ. (The precise definition of $\tilde{V}_{\omega,\Lambda}$ is given in Sect. 2.) The contribution from the second term in the right-hand side of (H.11) is

$$\frac{1}{2\pi i}\int dz\,\mathrm{Tr}\,\chi_\Omega R_\Lambda(z)\tilde{W}(\chi_\Lambda^\delta)R(z)x\chi_\Lambda^\delta P_{\mathrm{F}}y\chi_\Lambda^\delta P_{\mathrm{F}}\chi_\Omega.\tag{H.12}$$

The integrand is estimated by

$$\sum_{\mathbf{u}:s_b(\mathbf{u})\cap\Omega\ne\emptyset}\ \sum_{\mathbf{v},\mathbf{w}}|\,\mathrm{Tr}\,\chi_b(\mathbf{u})R_\Lambda(z)\tilde{W}(\chi_\Lambda^\delta)R(z)x\chi_\Lambda^\delta\chi_b(\mathbf{v})P_{\mathrm{F}}\chi_b(\mathbf{w})y\chi_\Lambda^\delta P_{\mathrm{F}}\chi_b(\mathbf{u})|.\tag{H.13}$$

For any bounded operators A,B,

$$\begin{aligned}
|\,\mathrm{Tr}\,A\chi_b(\mathbf{v})P_{\mathrm{F}}\chi_b(\mathbf{w})B|&\le\sqrt{\mathrm{Tr}\,A\chi_b(\mathbf{v})P_{\mathrm{F}}\chi_b(\mathbf{v})A^*}\cdot\sqrt{\mathrm{Tr}\,B^*\chi_b(\mathbf{w})P_{\mathrm{F}}\chi_b(\mathbf{w})B}\\
&\le\mathrm{Const}\times\|A\|\|B\|,
\end{aligned}\tag{H.14}$$

where we have used the bound (8.27). Using this inequality, one has

$$\begin{aligned}
&|\,\mathrm{Tr}\,\chi_b(\mathbf{u})R_\Lambda(z)\tilde{W}(\chi_\Lambda^\delta)R(z)x\chi_\Lambda^\delta\chi_b(\mathbf{v})P_{\mathrm{F}}\chi_b(\mathbf{w})y\chi_\Lambda^\delta P_{\mathrm{F}}\chi_b(\mathbf{u})|\\
&\quad\le\mathrm{Const}\times\|\chi_b(\mathbf{u})R_\Lambda(z)\tilde{W}(\chi_\Lambda^\delta)R(z)x\chi_\Lambda^\delta\chi_b(\mathbf{v})\|\|\chi_b(\mathbf{w})y\chi_\Lambda^\delta P_{\mathrm{F}}\chi_b(\mathbf{u})\|
\end{aligned}$$

[12] The bound (8.31) of Lemma 8.4 holds also for $\Lambda=\mathbf{R}^2$.

$$\leq \text{Const} \times \sum_{\mathbf{u}'} L^2 \|\chi_b(\mathbf{u}) R_\Lambda(z) \tilde{\chi}_b(\mathbf{u}')\| \|\tilde{W}(\chi_\Lambda^\delta)\chi_b^\delta(\mathbf{u}') R(z) \chi_b(\mathbf{v})\|$$
$$\times \|\chi_b(\mathbf{w}) P_{\mathrm{F}} \chi_b(\mathbf{u})\|, \tag{H.15}$$

where the sum is over $\mathbf{u}'$ such that $\operatorname{supp} \chi_b^\delta(\mathbf{u}') \cap \operatorname{supp} |\nabla \chi_\Lambda^\delta| \neq \emptyset$. Because of the existence of the indicator function $\mathbf{I}(M_\Lambda)$ in (H.6), the first factor in the summand is estimated as

$$\|\chi_b(\mathbf{u}) R_\Lambda(z) \tilde{\chi}_b(\mathbf{u}')\| \mathbf{I}(M_\Lambda) \leq \text{Const} \times L^{\kappa(\xi-2)+4} \exp[-\mu_\infty L^{2\kappa/3}] \tag{H.16}$$

from Lemma 8.1. The second factor can be estimated by using Lemma H.1. Therefore it is sufficient to estimate the following quantity near the Fermi energy:

$$\mathbf{E} \int_{y_-}^{y_+} dy \|\tilde{\chi}_b(\mathbf{u}') R(E_{\mathrm{F}} + iy) \chi_b(\mathbf{v})\| \|\chi_b(\mathbf{w}) P_{\mathrm{F}} \chi_b(\mathbf{u})\|. \tag{H.17}$$

Note that

$$\int_{y_-}^{y_+} dy \|\tilde{\chi}_b(\mathbf{u}') R(E_{\mathrm{F}} + iy) \chi_b(\mathbf{v})\| \leq \int_{y_-}^{y_+} dy \|\tilde{\chi}_b(\mathbf{u}') R(E_{\mathrm{F}} + iy) \chi_b(\mathbf{v})\|^{s/2} |y|^{s/2-1}. \tag{H.18}$$

Substituting this into the above, we obtain

$$\mathbf{E} \int_{y_-}^{y_+} dy \|\tilde{\chi}_b(\mathbf{u}') R(E_{\mathrm{F}} + iy) \chi_b(\mathbf{v})\| \|\chi_b(\mathbf{w}) P_{\mathrm{F}} \chi_b(\mathbf{u})\|$$
$$\leq \liminf_{\varepsilon_n \to 0} \int_{I_n} dy |y|^{s/2-1} \mathbf{E}[\|\tilde{\chi}_b(\mathbf{u}') R(E_{\mathrm{F}} + iy) \chi_b(\mathbf{v})\|^{s/2} \|\chi_b(\mathbf{w}) P_{\mathrm{F}} \chi_b(\mathbf{u})\|]$$
$$\leq \liminf_{\varepsilon_n \to 0} \int_{I_n} dy |y|^{s/2-1} \mathbf{E}[\|\tilde{\chi}_b(\mathbf{u}') R(E_{\mathrm{F}} + iy) \chi_b(\mathbf{v})\|^{s}]^{1/2} \mathbf{E}[\|\chi_b(\mathbf{w}) P_{\mathrm{F}} \chi_b(\mathbf{u})\|^2]^{1/2}$$
$$\leq \text{Const} \times \exp[-\mu |\mathbf{u}' - \mathbf{v}|/2] \exp[-\mu |\mathbf{w} - \mathbf{u}|/2], \tag{H.19}$$

where we have written $I_n = [y_-, y_+] \backslash (-\varepsilon_n, \varepsilon_n)$, and we have used Fatou's lemma, Fubini–Tonelli theorem, Schwarz's inequality, the bounds (7.1) and (7.17). Thus, the corresponding contribution is vanishing as $L \uparrow \infty$.

Consequently, it is enough to consider $\operatorname{Tr} \chi_\Omega P_{\mathrm{F},\Lambda} x (\chi_\Lambda^\delta)^2 P_{\mathrm{F}} y \chi_\Lambda^\delta P_{\mathrm{F}} \chi_\Omega$ which comes from the first term in the right-hand side of (H.11). Using the adjoint of the geometric resolvent equation, $R(z)\chi_\Lambda^\delta = \chi_\Lambda^\delta R_\Lambda(z) - R(z)\tilde{W}(\chi_\Lambda^\delta) R_\Lambda(z)$, we have

$$\operatorname{Tr} \chi_\Omega P_{\mathrm{F},\Lambda} x (\chi_\Lambda^\delta)^2 P_{\mathrm{F}} y \chi_\Lambda^\delta P_{\mathrm{F}} \chi_\Omega = \operatorname{Tr} \chi_\Omega P_{\mathrm{F},\Lambda} x (\chi_\Lambda^\delta)^2 P_{\mathrm{F}} y (\chi_\Lambda^\delta)^2 P_{\mathrm{F},\Lambda} \chi_\Omega + \text{correction} \tag{H.20}$$

in the same way. The correction is vanishing as $L \uparrow \infty$. Using the geometric resolvent equation again, the first term in the right-hand side is written

$$\operatorname{Tr} \chi_\Omega P_{\mathrm{F},\Lambda} x (\chi_\Lambda^\delta)^2 P_{\mathrm{F}} y (\chi_\Lambda^\delta)^2 P_{\mathrm{F},\Lambda} \chi_\Omega$$
$$= \operatorname{Tr} \chi_\Omega P_{\mathrm{F},\Lambda} x \chi_\Lambda^\delta P_{\mathrm{F},\Lambda} y (\chi_\Lambda^\delta)^3 P_{\mathrm{F},\Lambda} \chi_\Omega$$
$$+ \frac{1}{2\pi i} \int_\gamma dz \operatorname{Tr} \chi_\Omega P_{\mathrm{F},\Lambda} x \chi_\Lambda^\delta R_\Lambda(z) \tilde{W}(\chi_\Lambda^\delta) R(z) y (\chi_\Lambda^\delta)^2 P_{\mathrm{F},\Lambda} \chi_\Omega. \tag{H.21}$$

The integrand in the second term in the right-hand side is written

$$\begin{aligned}
&\mathrm{Tr}\,\chi_\Omega P_{\mathrm{F},\Lambda} x\chi_\Lambda^\delta R_\Lambda(z)\tilde{W}(\chi_\Lambda^\delta)R(z)y(\chi_\Lambda^\delta)^2 P_{\mathrm{F},\Lambda}\chi_\Omega \\
&\quad = \mathrm{Tr}\,\chi_\Omega P_{\mathrm{F},\Lambda} x\chi_\Lambda^\delta \chi_{\Lambda'} R_\Lambda(z)\tilde{W}(\chi_\Lambda^\delta)R(z)y(\chi_\Lambda^\delta)^2 P_{\mathrm{F},\Lambda}\chi_\Omega \\
&\qquad + \mathrm{Tr}\,\chi_\Omega P_{\mathrm{F},\Lambda} x\chi_\Lambda^\delta (1-\chi_{\Lambda'}) R_\Lambda(z)\tilde{W}(\chi_\Lambda^\delta)R(z)y(\chi_\Lambda^\delta)^2 P_{\mathrm{F},\Lambda}\chi_\Omega,
\end{aligned} \tag{H.22}$$

where $\chi_{\Lambda'}$ is the characteristic function of the region Λ' which satisfies the conditions of (8.4). This right-hand side can be shown to be vanishing as $L \uparrow \infty$ in the same way. Consequently, we obtain

$$\mathrm{Tr}\,\chi_\Omega P_{\mathrm{F},\Lambda} x\chi_\Lambda^\delta P_{\mathrm{F},\Lambda} y(\chi_\Lambda^\delta)^3 P_{\mathrm{F},\Lambda}\chi_\Omega = \mathrm{Tr}\,\chi_\Omega P_{\mathrm{F},\Lambda} x P_{\mathrm{F},\Lambda} y P_{\mathrm{F},\Lambda}\chi_\Omega + \text{correction}. \tag{H.23}$$

The first term in the right-hand side is nothing but the desired form. □

Appendix I: The Index Formula for the Switch Functions

The aim of this appendix is to give a proof of the following theorem:

Theorem I.1 *For a fixed period ℓ_P of the potentials $\mathbf{A}^{\mathrm{LP}}$ and V_0^{LP} in the Hamiltonian H_ω^{LP} of* (9.13) *on the whole plane $\mathbf{R}^2$, the following relation is valid almost surely*:

$$\mathrm{Index}(P_{\mathrm{F}} U_{\mathbf{a}} P_{\mathrm{F}}) = 2\pi i\, \mathrm{Tr}\, P_{\mathrm{F}}[[P_{\mathrm{F}}, \lambda_{1,\mathbf{a}}], [P_{\mathrm{F}}, \lambda_{2,\mathbf{a}}]], \tag{I.1}$$

where $\lambda_{j,\mathbf{a}}$, $j = 1, 2$, are two switch functions given by

$$\begin{aligned}
\lambda_{1,\mathbf{a}}(\mathbf{r}) &:= \begin{cases} 1, & \textit{for } x - a_1 \ge 0; \\ 0, & \textit{for } x - a_1 < 0, \end{cases} \\
\lambda_{2,\mathbf{a}}(\mathbf{r}) &:= \begin{cases} 1, & \textit{for } y - a_2 \ge 0; \\ 0, & \textit{for } y - a_2 < 0 \end{cases}
\end{aligned} \tag{I.2}$$

with the locations $\mathbf{a} = (a_1, a_2) \in \mathbf{R}^2$ of the steps.

Remark

1. The right-hand side of (I.1) is equal to the form of another Hall conductance which was discussed in [10, 11]. Elgart and Schlein [7] justified this Hall conductance formula within the linear response approximation under the assumption that the Fermi energy lies in a spectral gap. They also proved that the value of (I.1) takes the desired integer under the same gap assumption. As mentioned above, Germinet, Klein and Schenker proved the constancy of (I.1) in the localization regime, for a random Landau Hamiltonian with translation ergodicity, by using a consequence of the multiscale analysis.
2. From Theorem 9.4, we obtain that the Hall conductance using the position operator is equal to that using the switch functions.

We write the index as $\mathcal{I}_s(P_{\mathrm{F}};\ell_P)=2\pi i\,\mathrm{Tr}\,P_{\mathrm{F}}[[P_{\mathrm{F}},\lambda_{1,\mathbf{a}}],[P_{\mathrm{F}},\lambda_{2,\mathbf{a}}]]$. First, we shall show that the index $\mathcal{I}_s(P_{\mathrm{F}};\ell_P)$ is well defined for almost every ω. Note that

$$\begin{aligned}
&\mathrm{Tr}\,|[P_{\mathrm{F}},\lambda_{1,\mathbf{a}}][P_{\mathrm{F}},\lambda_{2,\mathbf{a}}]|\\
&\quad\le\sum_{\mathbf{u},\mathbf{v},\mathbf{w}\in(\mathbf{Z}_\varepsilon^2)^*}\mathrm{Tr}\,|\chi_\varepsilon(\mathbf{u})[P_{\mathrm{F}},\lambda_{1,\mathbf{a}}]\chi_\varepsilon(\mathbf{v})[P_{\mathrm{F}},\lambda_{2,\mathbf{a}}]\chi_\varepsilon(\mathbf{w})|\\
&\quad\le\sum_{\mathbf{u},\mathbf{v},\mathbf{w}\in(\mathbf{Z}_\varepsilon^2)^*}|\lambda_{1,\mathbf{a}}(\mathbf{u})-\lambda_{1,\mathbf{a}}(\mathbf{v})||\lambda_{2,\mathbf{a}}(\mathbf{v})-\lambda_{2,\mathbf{a}}(\mathbf{w})|\,\mathrm{Tr}\,|\chi_\varepsilon(\mathbf{u})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{v})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{w})|,
\end{aligned}\tag{I.3}$$

where $\chi_\varepsilon(\mathbf{u})$ is the characteristic function of the $\varepsilon_1\times\varepsilon_2$ rectangular box $s_\varepsilon(\mathbf{u})$ centered at $\mathbf{u}$, and we have chosen the set $(\mathbf{Z}_\varepsilon^2)^*$ of the centers $\mathbf{u}$ of the boxes $s_\varepsilon(\mathbf{u})$ so that $\mathbf{a}$ becomes a vertex of a rectangular box, i.e., $\mathbf{a}\in\mathbf{Z}_\varepsilon^2$. Using Schwarz's inequality, we have

$$\begin{aligned}
&\mathbf{E}[\mathrm{Tr}\,|\chi_\varepsilon(\mathbf{u})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{v})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{w})|]\\
&\quad\le\sqrt{\mathbf{E}[\mathrm{Tr}\,\chi_\varepsilon(\mathbf{u})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{v})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{u})]}\sqrt{\mathbf{E}[\mathrm{Tr}\,\chi_\varepsilon(\mathbf{w})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{v})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{w})]}
\end{aligned}\tag{I.4}$$

and

$$\begin{aligned}
&\mathrm{Tr}\,\chi_\varepsilon(\mathbf{u})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{v})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{u})\\
&\quad\le\sqrt{\mathrm{Tr}\,\chi_\varepsilon(\mathbf{u})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{u})\cdot\mathrm{Tr}\,\chi_\varepsilon(\mathbf{u})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{v})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{v})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{u})}\\
&\quad\le\mathrm{Const}\times\|\chi_\varepsilon(\mathbf{u})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{v})\|,
\end{aligned}\tag{I.5}$$

where we have used the bound (8.27). From these bounds, we obtain

$$\begin{aligned}
&\mathbf{E}[\mathrm{Tr}\,|\chi_\varepsilon(\mathbf{u})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{v})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{w})|]\\
&\quad\le\mathrm{Const}\times\sqrt{\mathbf{E}[\|\chi_\varepsilon(\mathbf{u})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{v})\|]}\sqrt{\mathbf{E}[\|\chi_\varepsilon(\mathbf{w})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{v})\|]}\\
&\quad\le\mathrm{Const}\times e^{-\mu|\mathbf{u}-\mathbf{v}|/2}e^{-\mu|\mathbf{w}-\mathbf{v}|/2},
\end{aligned}\tag{I.6}$$

where we have used the decay bound (7.17) for the Fermi sea projection. Note that

$$|\lambda_{j,\mathbf{a}}(\mathbf{u})-\lambda_{j,\mathbf{a}}(\mathbf{v})|=\begin{cases}0, & \text{for } (u_j-a_j)(v_j-a_j)>0;\\ 1, & \text{for } (u_j-a_j)(v_j-a_j)<0,\end{cases}\tag{I.7}$$

and

$$\sqrt{|x_1-y_1|^2+|x_2-y_2|^2}\ge|x_1-y_1|/2+|x_2-y_2|/2.\tag{I.8}$$

Combining these, (I.3) and (I.6), we obtain

$$\begin{aligned}
\mathbf{E}[\mathrm{Tr}\,|[P_{\mathrm{F}},\lambda_{1,\mathbf{a}}][P_{\mathrm{F}},\lambda_{2,\mathbf{a}}]|]\le\mathrm{Const}\times&\sum_{\mathbf{u},\mathbf{v},\mathbf{w}}e^{-\mu|u_1-a_1|/4}e^{-\mu|v_1-a_1|/4}e^{-\mu|u_2-v_2|/4}\\
&\times e^{-\mu|w_2-a_2|/4}e^{-\mu|v_2-a_2|/4}e^{-\mu|w_1-v_1|/4}<\infty.
\end{aligned}\tag{I.9}$$

Thus the operator $[P_{\mathrm{F}},\lambda_{1,\mathbf{a}}][P_{\mathrm{F}},\lambda_{2,\mathbf{a}}]$ is trace class for almost every ω.

Next we show that the index $\mathcal{I}_s(P_{\rm F};\ell_P)$ is independent of the locations a_1, a_2 of the steps of the switch functions $\lambda_{j,\mathbf{a}}$. Let $\mathbf{a}', \mathbf{a} \in \mathbf{R}^2$. Then we have

$$\begin{aligned}
&\mathrm{Tr}\, P_{\rm F}[[P_{\rm F}, \lambda_{1,\mathbf{a}'}], [P_{\rm F}, \lambda_{2,\mathbf{a}'}]] - \mathrm{Tr}\, P_{\rm F}[[P_{\rm F}, \lambda_{1,\mathbf{a}}], [P_{\rm F}, \lambda_{2,\mathbf{a}}]] \\
&\quad = \mathrm{Tr}\, P_{\rm F}[[P_{\rm F}, (\lambda_{1,\mathbf{a}'} - \lambda_{1,\mathbf{a}})], [P_{\rm F}, \lambda_{2,\mathbf{a}'}]] \\
&\qquad + \mathrm{Tr}\, P_{\rm F}[[P_{\rm F}, \lambda_{1,\mathbf{a}}], [P_{\rm F}, (\lambda_{2,\mathbf{a}'} - \lambda_{2,\mathbf{a}})]]. \qquad \text{(I.10)}
\end{aligned}$$

We will prove that the first term in the right-hand side is vanishing because the second term can be handled in the same way. We choose $\varepsilon = (\varepsilon_1, \varepsilon_2)$ so that both of $\mathbf{a}'$ and $\mathbf{a}$ satisfy $\mathbf{a}', \mathbf{a} \in \mathbf{Z}_\varepsilon^2(\mathbf{b}) := \mathbf{Z}_\varepsilon^2 - \mathbf{b}$ with some $\mathbf{b} \in \mathbf{R}^2$. We denote by $(\mathbf{Z}_\varepsilon^2(\mathbf{b}))^*$ the dual lattice of $\mathbf{Z}_\varepsilon^2(\mathbf{b})$.

Lemma I.2 *For* $\mathbf{u}, \mathbf{v} \in (\mathbf{Z}_\varepsilon^2(\mathbf{b}))^*$, *the following bound is valid*:

$$\mathbf{E}[\mathrm{Tr}\, |\chi_\varepsilon(\mathbf{u}) P_{\rm F} \chi_\varepsilon(\mathbf{v})|] \le \mathrm{Const} \times e^{-\mu'|\mathbf{u}-\mathbf{v}|} \qquad \text{(I.11)}$$

with some positive constant μ'.

Proof Note that

$$\begin{aligned}
\mathbf{E}[\mathrm{Tr}\, |\chi_\varepsilon(\mathbf{u}) P_{\rm F} \chi_\varepsilon(\mathbf{v})|] &\le \sum_{\mathbf{w} \in (\mathbf{Z}_\varepsilon^2(\mathbf{b}))^*} \mathbf{E}[\mathrm{Tr}\, |\chi_\varepsilon(\mathbf{u}) P_{\rm F} \chi_\varepsilon(\mathbf{w}) \chi_\varepsilon(\mathbf{w}) P_{\rm F} \chi_\varepsilon(\mathbf{v})|] \\
&\le \sum_{\mathbf{w} \in (\mathbf{Z}_\varepsilon^2(\mathbf{b}))^*} \sqrt{\mathbf{E}[\mathrm{Tr}\, \chi_\varepsilon(\mathbf{u}) P_{\rm F} \chi_\varepsilon(\mathbf{w}) P_{\rm F} \chi_\varepsilon(\mathbf{u})]} \\
&\qquad \times \sqrt{\mathbf{E}[\mathrm{Tr}\, \chi_\varepsilon(\mathbf{v}) P_{\rm F} \chi_\varepsilon(\mathbf{w}) P_{\rm F} \chi_\varepsilon(\mathbf{v})]}. \qquad \text{(I.12)}
\end{aligned}$$

Further, we have

$$\begin{aligned}
&\mathrm{Tr}\, \chi_\varepsilon(\mathbf{u}) P_{\rm F} \chi_\varepsilon(\mathbf{w}) P_{\rm F} \chi_\varepsilon(\mathbf{u}) \\
&\quad \le \sqrt{\mathrm{Tr}\, \chi_\varepsilon(\mathbf{u}) P_{\rm F} \chi_\varepsilon(\mathbf{u})} \sqrt{\mathrm{Tr}\, \chi_\varepsilon(\mathbf{u}) P_{\rm F} \chi_\varepsilon(\mathbf{w}) P_{\rm F} \chi_\varepsilon(\mathbf{w}) P_{\rm F} \chi_\varepsilon(\mathbf{u})} \\
&\quad \le \mathrm{Const} \times \|\chi_\varepsilon(\mathbf{u}) P_{\rm F} \chi_\varepsilon(\mathbf{w})\|, \qquad \text{(I.13)}
\end{aligned}$$

where we have used the bound (8.27). Combining this, the decay bound (7.17) for the Fermi sea projection, (I.12), we obtain

$$\begin{aligned}
&\mathbf{E}[\mathrm{Tr}\, |\chi_\varepsilon(\mathbf{u}) P_{\rm F} \chi_\varepsilon(\mathbf{v})|] \\
&\quad \le \mathrm{Const} \times \sum_{\mathbf{w} \in (\mathbf{Z}_\varepsilon^2(\mathbf{b}))^*} e^{-\mu|\mathbf{u}-\mathbf{w}|/2} e^{-\mu|\mathbf{w}-\mathbf{v}|/2} \le \mathrm{Const} \times e^{-\mu'|\mathbf{u}-\mathbf{v}|}. \qquad \text{(I.14)}
\end{aligned}$$

□

Now let us consider the first term in the right-hand side of (I.10). We write $\Delta\lambda$ for $\lambda_{1,\mathbf{a}'} - \lambda_{1,\mathbf{a}}$ for short.

Lemma I.3 *We have*

$$\mathbf{E}[\mathrm{Tr}\, |\Delta\lambda [P_{\rm F}, \lambda_{2,\mathbf{a}'}]|] < \infty \quad \textit{and} \quad \mathbf{E}[\mathrm{Tr}\, |\Delta\lambda P_{\rm F} [P_{\rm F}, \lambda_{2,\mathbf{a}'}]|] < \infty. \qquad \text{(I.15)}$$

Proof Without loss of generality, we can assume $a_1' > a_1$. Then we obtain

$$\begin{aligned}
\mathbf{E}[\mathrm{Tr}\,|\Delta\lambda[P_{\mathrm{F}},\lambda_{2,\mathbf{a}'}]|] &\le \sum_{\mathbf{u},\mathbf{v}\in(\mathbf{Z}_\varepsilon^2(\mathbf{b}))^*} \mathbf{E}[\mathrm{Tr}\,|\Delta\lambda\chi_\varepsilon(\mathbf{u})[P_{\mathrm{F}},\lambda_{2,\mathbf{a}'}]\chi_\varepsilon(\mathbf{v})|] \\
&\le \sum_{\mathbf{u},\mathbf{v}\in(\mathbf{Z}_\varepsilon^2(\mathbf{b}))^*} |\lambda_{1,\mathbf{a}'}(\mathbf{u})-\lambda_{1,\mathbf{a}}(\mathbf{u})||\lambda_{2,\mathbf{a}'}(\mathbf{v})-\lambda_{2,\mathbf{a}'}(\mathbf{u})| \\
&\quad\times \mathbf{E}[\mathrm{Tr}\,|\chi_\varepsilon(\mathbf{u})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{v})|] \\
&\le \sum_{a_1<u_1<a_1',u_2}\sum_{v_1,v_2} e^{-\mu'|u_1-v_1|/2}e^{-\mu'|u_2-a_2'|/2}e^{-\mu'|v_2-a_2'|/2} < \infty, \qquad \text{(I.16)}
\end{aligned}$$

where we have used (I.7), (I.8) and Lemma I.2.

Similarly, we have

$$\begin{aligned}
\mathbf{E}[\mathrm{Tr}\,|\Delta\lambda P_{\mathrm{F}}[P_{\mathrm{F}},\lambda_{2,\mathbf{a}'}]|] &\le \sum_{\mathbf{u},\mathbf{v},\mathbf{w}} \mathbf{E}[\mathrm{Tr}\,|\Delta\lambda\chi_\varepsilon(\mathbf{u})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{v})[P_{\mathrm{F}},\lambda_{2,\mathbf{a}'}]\chi_\varepsilon(\mathbf{w})|] \\
&\le \sum_{\mathbf{u},\mathbf{v},\mathbf{w}} |\lambda_{1,\mathbf{a}'}(\mathbf{u})-\lambda_{1,\mathbf{a}}(\mathbf{u})||\lambda_{2,\mathbf{a}'}(\mathbf{w})-\lambda_{2,\mathbf{a}'}(\mathbf{v})| \\
&\quad\times \mathbf{E}[\mathrm{Tr}\,|\chi_\varepsilon(\mathbf{u})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{v})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{w})|] \\
&\le \mathrm{Const}\times\sum_{\mathbf{u},\mathbf{v},\mathbf{w}} |\lambda_{1,\mathbf{a}'}(\mathbf{u})-\lambda_{1,\mathbf{a}}(\mathbf{u})||\lambda_{2,\mathbf{a}'}(\mathbf{w})-\lambda_{2,\mathbf{a}'}(\mathbf{v})| \\
&\quad\times e^{-\mu|\mathbf{u}-\mathbf{v}|/2}e^{-\mu|\mathbf{w}-\mathbf{v}|/2} \\
&\le \mathrm{Const}\times\sum_{a_1<u_1<a_1',v_1,w_1} e^{-\mu|u_1-v_1|/4}e^{-\mu|v_1-w_1|/4} \\
&\quad\times\sum_{u_2,v_2,w_2} e^{-\mu|u_2-v_2|/4}e^{-\mu|v_2-a_2'|/4}e^{-\mu|w_2-a_2'|/4} < \infty, \qquad \text{(I.17)}
\end{aligned}$$

where we have used the bound (I.6). □

Relying on this Lemma I.3, we have

$$\begin{aligned}
&\mathrm{Tr}\,P_{\mathrm{F}}[[P_{\mathrm{F}},\Delta\lambda],[P_{\mathrm{F}},\lambda_{2,\mathbf{a}'}]] \\
&\quad= \mathrm{Tr}\,P_{\mathrm{F}}\Delta\lambda(1-P_{\mathrm{F}})[P_{\mathrm{F}},\lambda_{2,\mathbf{a}'}] + \mathrm{Tr}[P_{\mathrm{F}},\lambda_{2,\mathbf{a}'}](1-P_{\mathrm{F}})\Delta\lambda P_{\mathrm{F}}. \qquad \text{(I.18)}
\end{aligned}$$

Further, the first term in the right-hand side is written

$$\begin{aligned}
&\mathrm{Tr}\,P_{\mathrm{F}}\Delta\lambda(1-P_{\mathrm{F}})[P_{\mathrm{F}},\lambda_{2,\mathbf{a}'}] \\
&\quad= \mathrm{Tr}\,\Delta\lambda(1-P_{\mathrm{F}})[P_{\mathrm{F}},\lambda_{2,\mathbf{a}'}]P_{\mathrm{F}} = \mathrm{Tr}\,\Delta\lambda(1-P_{\mathrm{F}})[P_{\mathrm{F}},\lambda_{2,\mathbf{a}'}], \qquad \text{(I.19)}
\end{aligned}$$

where we have used $(1-P_{\mathrm{F}})[P_{\mathrm{F}},\lambda_{2,\mathbf{a}'}](1-P_{\mathrm{F}})=0$. The second term becomes

$$\begin{aligned}
\mathrm{Tr}[P_{\mathrm{F}},\lambda_{2,\mathbf{a}'}](1-P_{\mathrm{F}})\Delta\lambda P_{\mathrm{F}} &= \mathrm{Tr}[P_{\mathrm{F}},\lambda_{2,\mathbf{a}'}](1-P_{\mathrm{F}})\chi_{\mathrm{supp}\,\Delta\lambda}\Delta\lambda P_{\mathrm{F}} \\
&= \mathrm{Tr}\,\Delta\lambda P_{\mathrm{F}}[P_{\mathrm{F}},\lambda_{2,\mathbf{a}'}](1-P_{\mathrm{F}})\chi_{\mathrm{supp}\,\Delta\lambda} \\
&= \mathrm{Tr}\,\Delta\lambda P_{\mathrm{F}}[P_{\mathrm{F}},\lambda_{2,\mathbf{a}'}](1-P_{\mathrm{F}}) \\
&= \mathrm{Tr}\,\Delta\lambda P_{\mathrm{F}}[P_{\mathrm{F}},\lambda_{2,\mathbf{a}'}], \qquad \text{(I.20)}
\end{aligned}$$

where $\chi_{\mathrm{supp}\,\Delta\lambda}$ is the characteristic function of the support of $\Delta\lambda$, and we have used $P_{\mathrm{F}}[P_{\mathrm{F}}, \lambda_{2,\mathbf{a}'}]P_{\mathrm{F}} = 0$. As a result, we obtain

$$\mathrm{Tr}\, P_{\mathrm{F}}[[P_{\mathrm{F}}, \Delta\lambda], [P_{\mathrm{F}}, \lambda_{2,\mathbf{a}'}]] = \mathrm{Tr}\, \Delta\lambda[P_{\mathrm{F}}, \lambda_{2,\mathbf{a}'}]. \tag{I.21}$$

This right-hand side is decomposed into two parts as

$$\mathrm{Tr}\, \Delta\lambda[P_{\mathrm{F}}, \lambda_{2,\mathbf{a}'}] = \mathrm{Tr}\, \Delta\lambda\chi_\ell[P_{\mathrm{F}}, \lambda_{2,\mathbf{a}'}] + \mathrm{Tr}\, \Delta\lambda(1 - \chi_\ell)[P_{\mathrm{F}}, \lambda_{2,\mathbf{a}'}] \tag{I.22}$$

with the characteristic function χ_ℓ of the square box centered at $\mathbf{r} = 0$ with a sufficiently large sidelength ℓ. Since we have

$$\mathbf{E}[\mathrm{Tr}\,|\chi_\ell P_{\mathrm{F}}|] \le \sum_{\mathbf{v}} \mathbf{E}[\mathrm{Tr}\,|\chi_\ell P_{\mathrm{F}}\chi_\varepsilon(\mathbf{v})|] < \infty \tag{I.23}$$

from Lemma I.2, the first term in the right-hand side is vanishing by cyclicity of the trace. The second term can be evaluated in the same way as in the proof of Lemma I.3. In consequence, it vanishes as $\ell \uparrow \infty$. Thus we obtain $\mathrm{Tr}\, P_{\mathrm{F}}[[P_{\mathrm{F}}, \Delta\lambda], [P_{\mathrm{F}}, \lambda_{2,\mathbf{a}'}]] = 0$. Since the second term in the right-hand side of (I.10) can be handled in the same way, the index $\mathcal{I}_s(P_{\mathrm{F}}; \ell_P)$ is independent of the locations a_1, a_2 of the steps of the switch functions.

Using this property, the index is written

$$\mathcal{I}_s(P_{\mathrm{F}}; \ell_P) = \frac{2\pi i}{\mathcal{V}_\ell} \sum_{\mathbf{a}\in\Lambda_\ell} \sum_{\mathbf{u},\mathbf{v},\mathbf{w}\in(\mathbf{Z}_\varepsilon^2)^*} [D_{12}(\mathbf{v}, \mathbf{w}, \mathbf{u}; \mathbf{a}) - D_{21}(\mathbf{v}, \mathbf{w}, \mathbf{u}; \mathbf{a})] S(\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{u}) \tag{I.24}$$

with

$$D_{12}(\mathbf{v}, \mathbf{w}, \mathbf{u}; \mathbf{a}) := [\lambda_{1,\mathbf{a}}(\mathbf{v}) - \lambda_{1,\mathbf{a}}(\mathbf{w})][\lambda_{2,\mathbf{a}}(\mathbf{w}) - \lambda_{2,\mathbf{a}}(\mathbf{u})] \tag{I.25}$$

and

$$D_{21}(\mathbf{v}, \mathbf{w}, \mathbf{u}; \mathbf{a}) := [\lambda_{2,\mathbf{a}}(\mathbf{v}) - \lambda_{2,\mathbf{a}}(\mathbf{w})][\lambda_{1,\mathbf{a}}(\mathbf{w}) - \lambda_{1,\mathbf{a}}(\mathbf{u})], \tag{I.26}$$

where both Λ_ℓ and $\mathcal{V}_\ell$ are the same as in (9.18), and $S(\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{u})$ is given by (9.20). We also write

$$\begin{aligned} \mathcal{I}_s^\varepsilon(P_{\mathrm{F}}; \Omega, \ell_P) = \frac{2\pi i}{\mathcal{V}_\ell} \sum_{\mathbf{u}\in\Lambda_\ell^*} \sum_{\mathbf{v},\mathbf{w}\in(\mathbf{Z}_\varepsilon^2)^*} \sum_{\mathbf{a}\in\mathbf{Z}_\varepsilon^2} [D_{12}(\mathbf{v}, \mathbf{w}, \mathbf{u}; \mathbf{a}) - D_{21}(\mathbf{v}, \mathbf{w}, \mathbf{u}; \mathbf{a})] \\ \times S(\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{u}), \end{aligned} \tag{I.27}$$

where Λ_ℓ^* is given by (9.21).

Lemma I.4 *The following holds*: $\mathbf{E}[|\mathcal{I}_s(P_{\mathrm{F}}; \ell_P) - \mathcal{I}_s^\varepsilon(P_{\mathrm{F}}; \Omega, \ell_P)|] \to 0$ *as* $|\Omega| \uparrow \infty$.

Proof To begin with, we note that

$$\begin{aligned} |\mathbf{E}[\mathrm{Tr}\, \chi_\varepsilon(\mathbf{u}) P_{\mathrm{F}}\chi_\varepsilon(\mathbf{v}) P_{\mathrm{F}}\chi_\varepsilon(\mathbf{w}) P_{\mathrm{F}}\chi_\varepsilon(\mathbf{u})]| &\le \mathbf{E}[\mathrm{Tr}\,|\chi_\varepsilon(\mathbf{v}) P_{\mathrm{F}}\chi_\varepsilon(\mathbf{w}) P_{\mathrm{F}}\chi_\varepsilon(\mathbf{u})|] \\ &\le \mathrm{Const} \times e^{-\mu|\mathbf{v}-\mathbf{w}|/2} e^{-\mu|\mathbf{w}-\mathbf{u}|/2} \end{aligned} \tag{I.28}$$

which is derived from (I.6). Using this, (I.7) and (I.8), we have

$$\begin{aligned}
&\mathbf{E}[|\mathcal{I}_s(P_{\mathrm{F}};\ell_P)-\mathcal{I}_s^{\varepsilon}(P_{\mathrm{F}};\Omega,\ell_P)|]\\
&\quad\le\frac{\mathrm{Const}}{\ell^2}\sum_{\mathbf{a}\in\mathbf{Z}_\varepsilon^2\backslash\Lambda_\ell}\sum_{\mathbf{u}\in\Lambda_\ell^*}+\sum_{\mathbf{a}\in\Lambda_\ell}\sum_{\mathbf{u}\in(\mathbf{Z}_\varepsilon^2)^*\backslash\Lambda_\ell^*}\sum_{v_1,w_1}e^{-\mu|v_1-a_1|/4}e^{-\mu|w_1-a_1|/4}e^{-\mu|w_1-u_1|/4}\\
&\qquad\times\sum_{v_2,w_2}e^{-\mu|v_2-w_2|/4}e^{-\mu|w_2-a_2|/4}e^{-\mu|u_2-a_2|/4}\\
&\quad\le\frac{\mathrm{Const}}{\ell^2}\sum_{\mathbf{a}\in\mathbf{Z}_\varepsilon^2\backslash\Lambda_\ell}\sum_{\mathbf{u}\in\Lambda_\ell^*}+\sum_{\mathbf{a}\in\Lambda_\ell}\sum_{\mathbf{u}\in(\mathbf{Z}_\varepsilon^2)^*\backslash\Lambda_\ell^*}e^{-\mu'|u_1-a_1|}e^{-\mu|u_2-a_2|/4}
\end{aligned}\tag{I.29}$$

with a positive constant μ'. This right-hand side is easily shown to vanish as $\ell\uparrow\infty$. □

Using the identity, $\sum_{\mathbf{a}\in\mathbf{Z}_\varepsilon^2}[D_{12}(\mathbf{v},\mathbf{w},\mathbf{u};\mathbf{a})-D_{21}(\mathbf{v},\mathbf{w},\mathbf{u};\mathbf{a})]=-(\mathbf{v}-\mathbf{w})\times(\mathbf{w}-\mathbf{u})$, one has

$$\begin{aligned}
\mathcal{I}_s^{\varepsilon}(P_{\mathrm{F}};\Omega,\ell_P)&=-\frac{2\pi i}{\mathcal{V}_\ell}\sum_{\mathbf{u}\in\Lambda_\ell^*}\sum_{\mathbf{v},\mathbf{w}}(\mathbf{v}-\mathbf{w})\times(\mathbf{w}-\mathbf{u})\,\mathrm{Tr}\,\chi_\varepsilon(\mathbf{u})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{v})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{w})P_{\mathrm{F}}\chi_\varepsilon(\mathbf{u})\\
&=\mathcal{I}^{\varepsilon}(P_{\mathrm{F}};\Omega,\ell_P),
\end{aligned}\tag{I.30}$$

where we have used the expression (9.23) of $\mathcal{I}^{\varepsilon}(P_{\mathrm{F}};\Omega,\ell_P)$ and (9.24). Combining this, (9.28) and Lemma I.4, we obtain Theorem I.1.

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